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TABLE OF CONTENTS

LOGIC AND FOUNDATIONS . . . . . p. 497
SET THEORY 499
COMBINATORIAL ANALYSIS 501
ORDER, LATTICES 501

CLASSICAL ALGEBRA 502
THEORY OF NUMBERS 503
FIELDS 509
ALGEBRAIC GEOMETRY 509
LINEAR ALGEBRA 501
ASSOCIATIVE RINGS AND ALGEBRAS 502
NON-ASSOCIATIVE RINGS AND ALGEBRAS 505
HOMOLOGICAL ALGEBRA 505

GROUPS AND GENERALIZATIONS 506
TOPOLOGICAL GROUPS AND LIE THEORY 510
MISCELLANEOUS TOPOLOGICAL ALGEBRA 511

FUNCTIONS OF REAL VARIABLES 512
MEASURE AND INTEGRATION 513
FUNCTIONS OF A COMPLEX VARIABLE 514
FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS 518

SPECIAL FUNCTIONS 522
ORDINARY DIFFERENTIAL EQUATIONS 525
PARTIAL DIFFERENTIAL EQUATIONS 532
POTENTIAL THEORY 535
FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS 539

SEQUENCES, SERIES, SUMMABILITY 540
APPROXIMATIONS AND EXPANSIONS 541
FOURIER ANALYSIS 542
INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS 545
INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS . 547

FUNCTIONAL ANALYSIS . . . . . p. 549
CALCULUS OF VARIATIONS 554

GEOMETRY 557
CONVEX SETS AND GEOMETRIC INEQUALITIES . . . 559

GENERAL TOPOLOGY, POINT SET THEORY 561
ALGEBRAIC TOPOLOGY 564
DIFFERENTIAL GEOMETRY, MANIFOLDS 567

PROBABILITY 576
STATISTICS 580

NUMERICAL METHODS 585
COMPUTING MACHINES 589

MECHANICS OF PARTICLES AND SYSTEMS 590
STATISTICAL THERMODYNAMICS AND MECHANICS . . 591
ELASTICITY, PLASTICITY 594
FLUID MECHANICS, ACOUSTICS 597
OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS . . 601
CLASSICAL THERMODYNAMICS, HEAT TRANSFER . . 602
QUANTUM MECHANICS 603
RELATIVITY 608
ASTRONOMY 610
GEOPHYSICS 610

OPERATIONS RESEARCH, ECONOMETRICS, GAMES . . 610
BIOLOGY AND SOCIOLOGY 612
INFORMATION AND COMMUNICATION THEORY . . . 613
SERVOMECHANISMS AND CONTROL 613

HISTORY AND BIOGRAPHY 614
MISCELLANEOUS 615

AUTHOR INDEX

Abe, E. ... 2709
Abian, Sabet ... 2765, 2766, 2767
Adachi, R. ... 3264, 3265
Ahn, Jae Koo ... 3005, 3024
Aigner, A. ... 2699
Akopyan, S. A. ... 2850
Albertoni, S. ... 3208
Alekseev, V. M. ... 2901
Alimov, Yu. I. ... 3284
Al'perin, M. M. ... 3223
Alterman, Z. ... 3111
Altman, M. ... 2979
Amitsur, S. A. ... 2675
Anosov, D. V. ... 3283
Appl, F. C. ... 3095
Armentrout, S. ... 2965
Aslanov, S. K. ... 3189
Asser, G. ... 2581
Asmus, E. F., Jr. ... 2670
Atkin, A. O. L. ... 2635
Attéia, M. ... 3147
Au, T. ... 3152
Aubert, K. E. ... 2918
Aumann, R. J. ... 3274
Auslander, L. ... 3015
Auslander, M. ... 2669
Baer, R. ... 2687
Bagi, B. B. ... 2943
Balk, M. B. ... 2733
Banaschewski, B. ... 2903
Barit, O. ... 3022
Barnotti, I. ... 2658, 2664, 2665
Bartholomew, D. J. ... 3067
Bartle, R. G. ... 2932
Bastiani, A. ... 2925a–b, 2896
Bateman, H. ... 2802
Baten, W. D. ... 3074
Baudoin-Cohier, S. ... 3029
Bell, P. O. ... 3199
Bellman, R. ... 3108
Bereis, R. ... 2993
Berezanskiĭ, Yu. M. ... 2774
Berezin, F. A. ... 3014
Berg, L. ... 2838
Bernardes, N. ... 3212
Bernhart, A. ... 2937
Berstein, I. ... 2972
Bhatt, Shri Nivas ... 2845
Bhonsle, B. R. ... 2759
Białynicki-Birula, I. ... 3216
Biot, M. A. ... 3165
Birkhoff, G. ... 3131
Blokh, V. I. ... 3139
Boas, R. P., Jr. ... 2740
Bockstein, M. = Bokštein, M.
Bogdan, R. ... 3117
Bokštein, M. ... 2680a–b, 2681
Bollini, C. G. ... 3217
Boothby, W. M. ... 2752
Borisov, Yu. F. ... 3032, 3033, 3034
Borisovič, Yu. G. ... 2930
Borsuk, K. ... 2973
Botteuna, O. ... 2861, 2959
Boumana, A. A. ... 3184
Bousquet, G. H. ... 3266
Brauner, H. ... 2993
Braunschweiger, C. C. ... 2898
Breiot, M. ... 2832
Briggs, W. E. ... 2629, 2630
Brout, R. ... 3135
Browder, F. E. ... 2819
Brown, A. B. ... 2765, 2766, 2767
Brun, V. ... 3292
Buchdahl, H. A. 3255, 3256, 3257
Buchsbaum, D. A. ... 2669
Budiansky, B. ... 3158
Bugaievski, E. ... 3117
Bursu, W. ... 2653
Butzer, P. L. ... 2874
Cabannes, H. ... 3191
Campbell, L. L. ... 2827
Caricato, G. ... 3210
Carleson, L. ... 3281
Carlitz, L. ... 2754, 2761
Carnap, R. ... 2578, 3037
Čech, E. ... 2962
Chang, C. C. ... 2576
Chang, H. M. ... 2790
Chang, Su-Cheng ... 2978, 2979
Chang, T. S. ... 3211
Chen, Kien-kwong ... 2747
Cheng, D. K. ... 3203
Chernoff, H. ... 3081
Cherry, I. ... 3108
Chin, Yuan-shun ... 2799
Ciliberto, C. ... 2934a–b
Climescu, A. ... 2605
Cohen, E. ... 2616
Cohen, H. ... 2710
Cohen, P. J. ... 3004
Cohn, P. M. ... 2702
Cohn, R. M. ... 2647
Collins, W. D. ... 3179
Connor, W. S. ... 3078
Constantinescu, P. = Konstantinescu, P.
Cooley, I. D. ... 3151
Coppel, W. A. ... 3123
Cordes, H. O. ... 2772
Costa de Beauregard, O. 3250a–b
Cox, C. P. ... 3075
Coz, M. ... 3035
Croisot, R. ... 2666, 2667, 2668
Cronin, J. ... 2787
Császár, Á. ... 2963, 2964
Cuesta, N. = Cuesta Dutari, N.
Cuesta Dutari, N. ... 2600, 2601, 2602a–b, 3305
Curtis, C. W. ... 2673
Czipszer, J. ... 2963
Daleckiĭ, Yu. L. ... 2926
Dalgarno, A. ... 3232, 3233
Darwin, J. H. ... 3061
David, Y. ... 2627
Davis, M. ... 2583
Dawson, D. F. ... 2842
Dean, W. R. ... 3169
Debever, R. ... 3197
Dehalu, M. ... 3073
Deleanu, A. ... 2975
Dembowski, P. ... 2952
Deprit, A. ... 2910
Derman, C. ... 3036
Derr, J. I. ... 3097
Derry, D. ... 2947

(*Continued on cover 3*)

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LOGIC AND FOUNDATIONS

See also 3247, 3287.

2575:

Tarski, Alfred. What is elementary geometry? Bol. Soc. Mat. Mexicana (2) **3** (1958), 41–51. (Spanish)

Tarski defines elementary geometry as that formalized within the first order predicate calculus. A system of geometry E_2 is proposed with variables ranging over points of the geometry and with betweenness and segment congruence as primitive predicates. It is shown that models of E_2 are isomorphic to a Cartesian space over some real closed field, that E_2 is complete, consistent, decidable, and not finitely axiomatizable. Two additional systems are considered. In the first, which is shown, with its consistent extensions, to be undecidable, variables over finite sets of points, the predicate of elementhood, and quantification over set variables are admitted in the statement of the axiom of continuity. In the second, which is decidable with respect to its universal propositions, the axiom of continuity of E_2 is replaced by an axiom stating that a segment joining a point inside a circle to a point outside always cuts the circle. *E. J. Cogan* (Bronxville, N.Y.)

2576:

Chang, C. C. On unions of chains of models. Proc. Amer. Math. Soc. **10** (1959), 120–127.

This paper contains results in the theory of models which were presented at the Amer. Math. Soc. meeting in August, 1956. The principal result is the following: "Let K be an arithmetical class (in the wider sense) of relational systems. Then K is the class of all models of some finite or infinite set of sentences of the form

$$\prod x_1, \cdots, x_n \sum y_1, \cdots, y_m \nu,$$

where ν is quantifier-free, if and only if K is closed under unions of chains of models." A second result is a strengthening of a result of A. Robinson [9.1.3. of *The metamathematics of algebra*, North-Holland Publishing Co., Amsterdam, 1951; MR **13**, 715] on convex algebras of axioms. The original proofs have been simplified by making use of results of Vaught [Tarski and Vaught, Compositio Math. **13** (1958), 81–102; MR **20** #1627]. {On page 122, line 7, the author clearly intends that "L be the class of all models of the negation of the sentence $\xi \cdots$".}

J. D. Rutledge (Yorktown Heights, N.Y.)

2577:

Martin, R. M. A formalization of inductive logic. J. Symb. Logic **23** (1958), 251–256.

Der Verfasser der vorliegenden Arbeit schlägt in einer kurzen Skizze einen exakten Aufbau von Carnaps "Theory of Confirmation" vor. Dabei ist hervorstechend, dass die diesem Aufbau zugrunde liegende Metasprache wesentlich einfacher ist, als nach Carnaps weniger formalen Ausführungen zu vermuten war. Die Frage nach weiteren Vereinfachungen bleibt offen; insbesondere, ob man die volle Theorie der reellen Zahlen vermeiden kann und ebenso die Verwendung von Variablen für Relationen.

{Es sei noch der folgende Druckfehler auf S.254, Zeile 6 v.u. vermerkt: Statt "an expression d" muss es heissen "an individual constant d".} *H. Kiesow* (Münster)

2578:

★**Carnap, Rudolf. Introduction to symbolic logic and its applications.** Translated by W. H. Meyer and J. Wilkinson. Dover Publications, Inc., New York, 1959. xiv + 241 pp. $1.85.

Translation, with some revisions, of the author's *Einführung in die symbolische Logik mit besonderer Berücksichtigung ihrer Anwendungen*, Springer, Vienna, 1954 [MR **16**, 208].

2579:

Moisil, Gr. C. Sur la logique positive. An. Univ. "C. I. Parhon" Bucureşti. Ser. Acta Logica **1** (1958), no. 1, 149–171. (Russian and English summaries)

The subject of the paper is the propositional calculus $P(\&, \vee)$ in & and $\vee$ only, formalized in an extended Gentzen calculus which can be described as follows. A schema of order 1 is a formula of $P(\&, \vee)$. If $S_1^{(n)}, \cdots, S_r^{(n)}, T^{(n)}$ are schemas of order n, then

$$\{S_1^{(n)}, \cdots, S_r^{(n)}\} \overrightarrow{\ }_{n+1} \{T^{(n)}\}$$

is a schema of order $n+1$. Here $\overrightarrow{\ }_n$ is a new sign for each value of n. For each n, Gentzen rules for permutation of premises (Perm_n), repetition of premises (Rep_n), addition of premises (Renf_n), extended modus ponens (Ench_n) and extraction (Extr_{n+1}) $[\{S_1^{(n)}, \cdots, S_r^{(n)}\} \overrightarrow{\ }_{n+1} \{S_i^{(n)}\}]$ are defined. The calculus of second order schemas C_2 has five obvious second order schemas $1_2, \cdots, 5_2$, as follows: $\{A \,\&\, B\} \overrightarrow{\ }_2 \{A\}$, etc., and the third order schemas

$$\mathrm{Conj}: \{(\Gamma) \overrightarrow{\ }_2 (A), (\Gamma) \overrightarrow{\ }_2 (B)\} \overrightarrow{\ }_3 \{(\Gamma) \overrightarrow{\ }_2 (A \,\&\, B)\},$$

$$\mathrm{Dil}: \{(A) \overrightarrow{\ }_2 (C), (B) \overrightarrow{\ }_2 (C)\} \overrightarrow{\ }_3 \{(A \vee B) \overrightarrow{\ }_2 (C)\},$$

where Γ is a finite sequence of first order schemas.

The definition of validity of a second order schema in C_2 is by recursion, starting with $1_2, \cdots, 5_2$ and operating by Perm_3, Rep_3, Renf_3, Conj, Dil, Ench_3. The calculus of third order schemas C_3 is defined by $1_3, \cdots, 5_3$ (analogous to $1_2, \cdots, 5_2$), Perm_3, Rep_3, Renf_3, Conj, Dil, Ench_3, Extr_3, Perm_4, Rep_4, Renf_4, Ench_4, Extr_4. Validity in C_3 is defined by recursion.

Theorem 1: If second order schemas $\Sigma_1, \cdots, \Sigma_r$ are valid in C_2 and if $(\Sigma_1, \cdots, \Sigma_r)\overrightarrow{\tau}(\Sigma)$ is valid in C_2, then Σ is valid in C_2. Theorem 2: Σ is valid in C_2 if and only if $\overrightarrow{\tau}(\Sigma)$ is valid in C_2.

Equivalence $A\overleftrightarrow{\tau}B$ is introduced, and it is proved that equivalence classes with respect to this relation form a lattice L. Theorem 7: $(A_1, \cdots, A_r)\overrightarrow{\tau}(B)$ is valid if and only if, for every lattice Λ and any substitution Σ of elements of Λ for the variables in $A_1, \cdots, A_r, B$,

$$A_1{}^0 \cap \cdots \cap A_r{}^0 \subset B^0,$$

where $A_1{}^0, \cdots, A_r{}^0, B^0$ are the values obtained from $A_1, \cdots, A_r, B$ by the substitution Σ. Theorem 8: The schemas of order 1, derivable from given schemas, form a principal ideal in L. Analogously: Let E be a set of schemas of order n; $S^{(n)} \in CE$ if there exist schemas $S_1{}^{(n)} \in E, \cdots, S_r{}^{(n)} \in E$ such that $(S_1{}^{(n)}, \cdots, S_r{}^{(n)})\overrightarrow{\tau}S^{(n)}$; then I: $E \subset CE$; II: if $E \subset F$, then $CE \subset CF$; III: $CCE = CE$; IV: if $E_t = CE_t$ and $E = \bigcap_t E_t$, then $E = CE$.

A. Heyting (Amsterdam)

2580:

Mihailescu, Eugen. Recherches sur quelques systèmes du calcul des propositions. An. Univ. "C. I. Parhon" Bucureşti. Ser. Acta Logica **1** (1958), no. 1, 173–185. (Russian and English summaries)

The author establishes a normal form for statements in each of the two systems of propositional logic that he considers. The first is based on equivalence and conjunction. The second is based on these two and on reciprocity, for which the additional axioms are *EERpqRrsEEpqErs* and *EREpqrRKprKqr*. *E. J. Cogan* (Bronxville, N.Y.)

2581:

Asser, Günter; und Schröter, Karl. Axiomatisierung der k-zahlig allgemeingültigen Ausdrücke des Stufenkalküls. Math. Nachr. **19** (1958), 73–86.

The authors consider the theory of types in which only one-place predicates occur. For a positive integer k, a formula is k-valid if and only if it is true in every model in which the domain of individuals contains exactly k elements. Clearly, the set of k-valid formulas is decidable, and therefore axiomatizable. This paper shows that a suitable axiomatization is obtained by adding to the axioms of type theory the assertion that there are exactly k individuals. The proof is based upon the reduction of universal quantification to a suitable finite conjunction and of existential quantification to a suitable finite disjunction. The authors also point out that every model of the set of k-valid formulas is, after reduction, by means of equivalence classes, of the equality relation to the identity relation, just the standard k-model (in which there are k individuals and the $(n+1)$st type is the set of all subsets of the nth type). *E. Mendelson* (New York, N.Y.)

2582:

Fraïssé, Roland. Un modèle définissant une théorie aberrante des ensembles où sont niés les axiomes du choix et d'extensionalité. Publ. Sci. Univ. Alger. Sér. A **5** (1958), 17–98.

Fraenkel [S.-B. Preuss. Akad. Wiss. **1922**, 253–257] and Mostowski [Fund. Math. **32** (1939), 201–252] have constructed set-theories in which there exists an infinity of "atoms" (sets which have no members) and for which the axiom of choice is false. This is however the first detailed study of any model of such an anomalous set-theory. A model is here formed by starting from a set-theory which satisfies the Bernays-Gödel axioms and singling out some set e whose elements will be taken as the atoms which generate the new universe $\mathcal{E}$ of the anomalous model. We are here interested in those sets e which are infinite and moreover have the following property K*: If $a \in \mathcal{E}$ and $u \in e$, then $a \notin u$. Let us now define the rank α of an e-set (an element of $\mathcal{E}$) and e-φ sets. a is an e-set of rank α (α an ordinal) if $\alpha = 0$ and $a \in e$, or if $\alpha \neq 0$ and a is a set of elements each of which is an e-set of rank $\alpha' < \alpha$, but a is not itself an e-set of rank $\alpha' < \alpha$. A set a is an e-φ set of rank 0 if $a \in e$; an e-set a of rank $\alpha \neq 0$ is an e-φ set if: (i) the elements of a are e-φ sets of rank $< \alpha$, (ii) there is a finite subset $f \subseteq e$ such that for every mapping t which interchanges 2 elements of $(e-f)$ and is the identity elsewhere, we have $\bar{t}(a) = a$ ($\bar{t}$ is the natural extension of t to e). There is a corresponding definition of e-φ classes. It is now shown in detail that if e is infinite and has the property K*, then the e-φ sets and classes form a model of a set-theory in which all Gödel's axioms hold except extensionality (axiom $A3$) and the axiom of choice (axiom E). The last section of the paper consists in the examination of some of the properties of such an anomalous set-theory and the derivation of various results which contradict the axiom of choice. For example we have the theorems: (a) the set e of atoms is infinite but cannot be divided into two disjoint infinite subsets; (b) e cannot be linearly ordered; (c) if $u \in e$, then $e-\{u\}$ cannot be put into 1-1 correspondence with e, nor $e \times e$ with e.

L. N. Gál (New Haven, Conn.)

2583:

Davis, Martin; and Putnam, Hilary. Reductions of Hilbert's tenth problem. J. Symb. Logic **23** (1958), 183–187.

Hilbert's problem is to find an algorithm for determining for every given diophantine equation whether it possesses solutions. A diophantine predicate is one obtained from a diophantine equation by existential quantification of some of its variables. It is well known that if it can be shown that some recursively enumerable non-recursive predicate is diophantine, the recursive unsolvability of Hilbert's problem will directly follow.

Notation: $R(x,y)$ is the remainder of x when divided by y;

$$(Ex_1, \cdots, x_n)^y_{a_1, \cdots, a_n}[\cdots],$$

means that there are exactly y n-tuples $(x_1, \cdots, x_n)$ with $x_1 \leqq a_1, \cdots, x_n \leqq a_n$ such that $[\cdots]$. The following results are proved. If the predicates

$$z = \sum_{i=1}^{n} R(a, 1+ig), \quad 0 = \sum_{i=1}^{y} R(a-i, 1+ig)$$

are diophantine, then every recursively enumerable predicate is diophantine. If the three predicates

$$P_1(a,b,c) \equiv (Ex,y,z)^z_{b,b,a}[(1+cz)x = y],$$
$$P_2(a,b,c) \equiv (Ex,z)^z_{b,a}[(1+cz)x = b-z],$$
$$P_3(a,b,c) \equiv (Ex,y,z)^z_{b,a,a}[(1+cz)x = b-y]$$

are diophantine, then every recursively enumerable predicate is diophantine. If the predicate

$$R(a,b,c,d,e,f,g) \equiv (Ex,y,z)^z_{f,g,a}[(1+cz)x = d+ey+gz]$$

is diophantine, then every recursively enumerable predicate is diophantine. These results are derived from a theorem of Martin Davis [same J. **18** (1953), 33–41; MR **14**, 1052] by number-theoretic arguments.

M. O. Rabin (Jerusalem)

2584:

Wang, Hao. Alternative proof of a theorem of Kleene. J. Symb. Logic **23** (1958), 250.

This paper provides a new proof of Theorem II of Kleene, Amer. J. Math. **77** (1955), 405–428 [MR **17**, 5]. The predicates $a <_O b$ and $a \in O$ are defined separately and the recursive predicates sought in the theorem are immediately at hand. The author remarks that, because of his classical use of the function quantifier, it is not clear whether the constructive content of Kleene's proof may by obtained by his method.

E. J. Cogan (Bronxville, N.Y.)

2585:

Rogers, Hartley, Jr. Gödel numberings of partial recursive functions. J. Symb. Logic **23** (1958), 331–341.

Starting from an intuitive concept of a Gödel numbering of partial recursive (p.r.) functions the author sets up the following formal definitions. The term "function" means, unless otherwise specified, function of one variable. Let ϕ_i $(i=0, 1, 2, \cdots)$ be a fixed enumeration of all p.r. functions which is obtained from Kleene's enumeration by a slight effective modification ensuring that every integer becomes the index of some function. This enumeration will be called the standard enumeration. A numbering is a mapping π of a recursive set D_π onto the set of all p.r. functions. A numbering π is called semi-effective if there exists a p.r. function of two variables Φ such that $i \in D_\pi$ implies $\pi i = \lambda x \Phi(i, x)$. We shall say that Φ describes π. A semi-effective numbering π is called fully effective if there exists a recursive function f with range in D_π such that, for all i, $\phi_i = \lambda x \Phi(f(i), x)$, where Φ describes π.

This last definition depends on a particular enumeration of p.r. functions. To obtain a characterization which is invariant, a partial ordering of numberings is introduced. A numbering ρ is derivable from a numbering π $(\rho \leqq \pi)$ if there exists a recursive function g mapping D_ρ into D_π such that $\rho = \pi g$ on D_ρ. Two numberings π and ρ are called equivalent if $\pi \leqq \rho$ and $\rho \leqq \pi$. It is proved that all fully effective numberings are equivalent to the standard enumerations (numbering). The fully effective numberings are also called Gödel numberings. The equivalence classes of semi-effective numberings form an upper semi-lattice under the relation $\leqq$. The class of Gödel numberings is now characterized as the unique maximal element of this lattice.

The main result concerns isomorphisms of Gödel numberings. Theorem: Given two Gödel numberings ρ and π, there exists a recursive function f which gives a 1-1 mapping of D_ρ onto D_π and satisfies $\rho = \pi f$ on D_ρ. The proof is obtained by a clever application of the recursion theorem. It is not known whether every two semi-effective numberings which are equivalent are isomorphic in the sense of the above theorem. It is a consequence of this theorem that all numberings arising from the different methods of defining p.r. functions (e.g., Post systems or Turing machines) are isomorphic. The paper concludes with an application of the theorem to the characterization of creative sets.

M. O. Rabin (Jerusalem)

2586:

Kleene, S. C. Quantification of number-theoretic functions. Compositio Math. **14**, 23–40 (1959).

In Section 1, the author shows that predicates expressible as $(E\alpha)(x)R(a, \alpha, x)$, with recursive R and the hyperarithmetical functions as a basis, are hyperarithmetical. Section 2 is devoted to a theorem giving as a function of $y \in O$ bases for the predicates $H_y(a)$ and $\overline{H}_y(a)$ expressed in the above form. Each basis consists of the functions each recursive in H_{w_0} for some $w_0 \leqq_O y$ with $|w_0|$ either equal to or less than a specified ordinal. In Section 3, the ramified analytical hierarchy of functions and predicates is defined and studied. Two kinds of analytical hierarchies are considered. It is shown that to express predicates in either hierarchy, it is sufficient to use, besides number quantifiers, a single function (one place predicate) quantifier, either existential or universal, with its range as a basis. For notation see Kleene [Bull. Amer. Math. Soc. **61** (1955), 193–213; MR **17**, 4].

E. J. Cogan (Bronxville, N.Y.)

2587:

★Schmidt, H. Arnold. Un procédé maniable de décision pour la logique propositionnelle intuitionniste. Le raisonnement en mathématiques et en sciences expérimentales, pp. 57–66. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

Decision procedures for the intuitionistic and the minimal propositional calculi in which the logical connectives conjunction, disjunction, implication and negation appear is presented. The method appears to be faster than the existing methods.

In the discussion A. Mostowski mentions a decision procedure discovered by Rasiowa and Sikorski for formulas in prenex normal form of the intuitionistic predicate calculus.

P. C. Gilmore (Yorktown Heights, N.Y.)

2588:

★Heyting, A. La théorie élémentaire de l'intégration en mathématiques intuitionnistes. Le raisonnement en mathématiques et en sciences expérimentales, pp. 85–90. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

In order to illustrate the differences between intuitionistic and classical methods of reasoning and results some definitions, theorems and proofs are given concerning the intuitionistic theory of integration.

P. C. Gilmore (Yorktown Heights, N.Y.)

2589:

★Heyting, A. Some remarks on intuitionism. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 69–71. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

It is necessary to distinguish between theories of the constructible and constructive theories; the main difference being that in a constructive theory an object exists

only after it has been constructed while in theories of the constructible one defines a certain class of mathematical objects to be constructible objects.

P. C. Gilmore (Yorktown Heights, N.Y.)

2590:

★**Kleene, S. C. Realizability.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 285–289. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

A notion of "realizability" was earlier used [S. C. Kleene, *Introduction to metamathematics*, North Holland Publ. Co., Amsterdam; van Nostrand, New-York Toronto; 1952; MR **14**, 525] for the study of intuitionistic arithmetic. Here a formal system of intuitionistic analysis is given together with a definition of realizability under which every provable formula is realizable.

P. C. Gilmore (Yorktown Heights, N.Y.)

2591:

Kreisel, G. Elementary completeness properties of intuitionistic logic with a note on negations of prenex formulae. J. Symb. Logic **23** (1958), 317–330.

The proofs in this paper are intuitionistic; therefore the following notions of completeness of a formal system F of the predicate calculus must be distinguished. Let $\mathfrak{A}$ be a formula of F containing the predicate symbols $P_1, \cdots, P_k$. (1) Strong completeness: If for all predicates $P_1^*, \cdots, P_k^*$, defined on some species S, $\mathfrak{A}(P_1^*, \cdots, P_k^*)$ holds, then $\mathfrak{A}$ is provable in F. (2) Weak completeness: If $\mathfrak{A}$ is not provable in F, then it is absurd that, for all $P_1^*, \cdots, P_k^*$, $\mathfrak{A}(P_1^*, \cdots, P_k^*)$ holds. (3) Strong completeness by substitution: For each $\mathfrak{A}$, there are predicates $P_1^{\mathfrak{A}}, \cdots, P_k^{\mathfrak{A}}$ such that, if $\mathfrak{A}(P_1^{\mathfrak{A}}, \cdots, P_k^{\mathfrak{A}})$ holds, then $\mathfrak{A}$ is provable in F. (4) Weak completeness by substitution: For each $\mathfrak{A}$ there are predicates $P_1^{\mathfrak{A}}, \cdots, P_k^{\mathfrak{A}}$ such that, if $\mathfrak{A}$ is not provable in F, then it is absurd that $\mathfrak{A}(P_1^{\mathfrak{A}}, \cdots, P_k^{\mathfrak{A}})$. (3)→(1) and (4)→(2), but not (1)→(4), as is illustrated by F = propositional calculus and $\mathfrak{A} = P \vee \neg P$. However, if for each primitive recursive predicate $A(x)$, $\neg(x)A(x) \to (Ex)\neg A(x)$ (it is not known whether this is true), then (2)→(1) and (4)→(3), provided the proof predicate for F is primitive recursive.—The fragment of the predicate calculus obtained by using only the connectives &, $\neg$ and the universal quantifier is weakly complete by substitution; the same result holds for the fragment consisting of the formulae in which a double negation occurs before every prime formula. On the other hand, the fragment which consists of negations of prenex formulae cannot be proved to be weakly complete by substitution by methods formalized in intuitionistic arithmetic, including primitive recursive functions. The proof of the latter result utilizes Mostowski's example of a formula which does not possess a recursive model, though it can be proved to be consistent by the methods of intuitionistic arithmetic. By means of Herbrand's theorem it is shown that the negation of a prenex formula is provable intuitionistically if and only if it is provable classically. Finally, for negations of prenex formulae a Herbrand theorem of the following form holds: With every such formula $\mathfrak{A}$ a sequence $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ of quantifier-free formulae can be associated so that: (1) If $\mathfrak{A}$ is provable, some $\mathfrak{A}_n$ is provable in the intuitionistic propositional calculus; (2) for each n, $\mathfrak{A}$ can be proved from $\mathfrak{A}_n$ in the intuitionistic predicate calculus; (3) if $\mathfrak{A} \to \mathfrak{B}$, then for each n there is an m such that $\mathfrak{B}_m$ can be proved from $\mathfrak{A}_n$ in the intuitionistic propositional calculus.

A. Heyting (Amsterdam)

SET THEORY

2592:

Marcus, Solomon. Sur les ensembles indépendants dans la théorie des relations. Monatsh. Math. **63** (1959), 244–255.

New proofs of old results, as well as proofs of new results in the theory of sets of independent elements. Sample results: (1) If, for each real number x, the sets $S(x)$ and $S^{-1}(x)$ are denumerable, then there exists an independent set which meets every perfect set. (2) Assuming that $2^{\aleph_0} = \aleph_1$, if for each real number x, with the possible exception of a set of the first category, the sets $S(x)$ and $S^{-1}(x)$ are of the first category; then there exists an independent set of the second category.

S. Ginsburg (Gardena, Calif.)

2593:

Ščegol'kov, E. A. On the uniformization and splitting of certain sets. Dokl. Akad. Nauk SSSR **124** (1959), 783–785. (Russian)

Various new cases of the uniformization of plane sets are announced. Any plane A_2-set [$\bar{A}_2$-set] S such that its intersection $S(x)$ with each line $x = \text{const}$ is closed and the projection $\mathrm{Pr}_x S$ on the x-axis is a B_2-set [$\bar{B}_2$-set] is uniformizable by means of a B_2-set [$\bar{B}_2$-set] (theorems 1 and 2 respectively). There exists a plane CA_2-set S such that $S(x)$ is finite and $\mathrm{Pr}_x S = [0,1]$, which is uniformizable neither with any B_2-set nor with any A_2-set (th. 3). For almost every n, any A_n-set S such that $S(x)$ is an F-set and $\mathrm{proj}_x S \in B_n$ is B_n-uniformizable. (This statement, theorem 4, is not contradictory in the set axiomatics of Gödel.) Problem of P. S. Novikov: If $S \in B$ and $\forall_x S(x) \in F_\sigma$, is then S the union of a countable sequence of B-sets P such that $\forall_x P(x) \in F$? The author resolves this problem in the affirmative for a special case and conjectures that the answer in every case should be yes. (The particular case resolved is that S is the projection of a spatial G_δ-set Q with "bounded regularity indices in each point" of the projection Q_x of Q on x-axis.)

(Every projective set $\mathcal{E}$ of class 2 of the line is the projection of a uniform plane CA-set E; the projections of the constituents of E yield a partition of $\mathcal{E}$ into a Ω-sequence E^β of B-sets; $\mathcal{E}$ is said to be of the class $\bar{A}_1$ provided for every completely additive measure F and every perfect set P one has a $\gamma < \Omega$ such that

$$F(\mathrm{Pr}_x \bigcup_{\beta > \gamma} E^\beta) = 0$$

or $\mathrm{Pr}_x (\bigcup_{\beta > \gamma} E^\beta)$ is of the first category on P; the class $\bar{B}_2$ is the intersection of $\bar{A}_2$ and $C\bar{A}_2$ [cf. A. A. Lyapunov, Mat. Sb. (N.S.) **20** (**62**) (1947), 179–196; MR **9**, 83].)

Đ. Kurepa (Princeton, N.J.)

COMBINATORIAL ANALYSIS

See also 2753.

2594:

★Mamuzić, Zlatko P. Kombinatorika. [Combinatorial analysis.] Matematička Biblioteka, 6. Univerzitet u Beogradu. Belgrade, 1957. 122 pp.

An introductory treatment in readable style.

2595:

Fort, M. K., Jr.; and Hedlund, G. A. Minimal coverings of pairs by triples. Pacific J. Math. 8 (1958), 709–719.

The authors show that the number of triples of distinct integers from $\{1, 2, \cdots, n\}$ containing each pair of distinct integers from $\{1, 2, \cdots, n\}$ at least once, is at least $n\lfloor n/2\rfloor/3$, and prove that the implicit minimum is attained for all $n \geq 3$. For $n \equiv 1, 3 \pmod 6$ all pairs in a minimal set appear just once and this is the Steiner triple system. For $n \equiv 5 \pmod 6$ all but one pair appear just once and that pair is in three triples. For $n \equiv 0 \pmod 6$ all but $n/2$ pairs, and for $n \equiv 2, 4 \pmod 6$ all but $n/2+1$ pairs, appear just once with the others appearing twice. Finally they show that any minimal set of triples for n is contained in some minimal set of triples for $2n$.

K. Goldberg (Washington, D.C.)

2596:

Lockwood, E. H.; and Gant, P. Factors of binomial coefficients. Math. Gaz. **43** (1959), 31–34.

The authors determine when the binomial coefficient ${}_nC_r$ is divisible by a prime p, in terms of the p-adic expansions of n and r. They also find how many of the coefficients of $(x+y)^n$ are not divisible by p. Their method consists in counting the number of "symmetrical" selections of n things r at a time. Their results are contained in a note of the reviewer [Amer. Math. Monthly **54** (1947), 589–592; MR **9**, 331].

N. J. Fine (Princeton, N.J.)

2597:

Gould, H. W. Note on a paper of Sparre Andersen. Math. Scand. **6** (1958), 226–230.

Two identities involving binomial coefficients given by Erik Sparre Andersen [Math. Scand. **1** (1953), 261–262; MR **15**, 387] are shown to be instances of the following identity

$$\sum_{k=0}^{n}\binom{x}{k}\Delta^k f(0) = (-1)^n\binom{x-1}{\alpha}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}\frac{x}{x-k}f(k)$$

with $\Delta f(n)=f(n+1)-f(n)$. {It may be noted that for $f(k)=(z-k)^{-1}$ this identity becomes

$$\sum_{k=0}^{n}\binom{x}{k}k!/(z)_{n+1} = (z-x)^{-1}[1-(x)_{n+1}/(z)_{n+1}],$$

with $(x)_n = x(x-1)\cdots(x-n+1)$, a result not completed in the paper.}

J. Riordan (New York, N.Y.)

2598:

Harary, Frank. On the number of bi-colored graphs. Pacific J. Math. **8** (1958), 743–755.

As understood in this paper, a graph is a finite set whose elements are called points, together with a set of unordered pairs of unequal points called lines. A graph is called k-colored, where k is an integer ≥ 1, if its points have been divided into k non-empty mutually disjoint (color) classes in such a way that no two points which are in the same class are joined by a line; if $k=2$ then the graph is called bi-colored.

The present paper is the fourth of a series by the author devoted to graph-counting problems. Its most important result is a counting polynomial for bi-colored graphs, which is obtained using the methods of Pólya. It is also shown how the counting polynomial for bi-colored graphs can be modified in order to enumerate all the different balanced signed graphs—these were treated by the author in Michigan Math. J. **2** (1953/54), 143–146 [MR **16**, 733]. In addition, the problem of enumerating 3-colored graphs is discussed, and also a formula is furnished from which the number of connected k-colored graphs with p points can be computed recursively in terms of the counting polynomial for all k-colored graphs.

G. A. Dirac (Hamburg)

2599:

Aigner, Alexander. Die Häufigkeitsverteilung gewisser Typen von endlichen Graphen. J. Reine Angew. Math. **200** (1958), 125–128.

Als Geradlösung eines endlichen Graphen definiert Verf. eine solche nichtleere Teilmenge seiner Punkte, zu der von jedem Punkt des gesamten Graphen aus eine gerade Anzahl von Kanten führt. In Graphen mit einer ungeraden Anzahl von Punkten gibt es mindestens eine Geradlösung [A. Aigner, dasselbe J. **195** (1955), 3–21; MR **17**, 464, 519]. Die Graphen verteilen sich nach der Anzahl ihrer unabhängigen Geradlösungen in Typen. Die Wahrscheinlichkeitsverteilung dieser Typen, wenn die Anzahl der Punkte ins Unendliche wächst, wird untersucht, unter Annahme der Kantenwahrscheinlichkeit $\frac{1}{2}$. Die relative Häufigkeit des Types n ist mit h_n bezeichnet, wobei $h_0, h_2, h_4, \cdots$ für Graphen mit gerader Punktezahl und $h_1, h_3, h_5, \cdots$ für Graphen mit ungerader Punktezahl zu verstehen sind. Dann ist

$$h_0 = \prod_{k=1}^{\infty}\frac{2^{2k-1}-1}{2^{2k-1}} = \prod_{k=1}^{\infty}\left(1+\frac{1}{2^k}\right)^{-1}; \quad h_l = h_0\prod_{i=1}^{l}\frac{2}{2^i-1}.$$

Die numerischen Werte sind: $h_0=0.4194$, $h_1=0.8388$, $h_2=0.5592$, $h_3=0.1598$, $h_4=0.0213$, $h_5=0.0014$; die folgenden liegen unter 0.0001. Der Anteil der höheren Typen bleibt also klein.

Der Erwartungswert der Gesamtzahl der Geradlösungen ergibt sich als 2 bei gerader Punktezahl und auch 2 bei ungerader Punktezahl.

G. A. Dirac (Hamburg)

ORDER, LATTICES

See also 2634.

2600:

Cuesta, N. On an article of Mac Neille. Rev. Mat. Hisp.-Amer. (4) **18** (1958), 3–9. (Spanish)

The author contends that Definition 11.1 of cut and Definition 11.5 of ordering of cuts given by MacNeille [Trans. Amer. Math. Soc. **42** (1937), 416–460] are, respectively, too restrictive and unnatural.

F. Bagemihl (Notre Dame, Ind.)

2601:

★**Cuesta Dutari, Norberto. Matematica del orden.** [Mathematics of order.] Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Tomos LII y LIII, Madrid, 1959. 513 pp.

The first two chapters have been reviewed [Rev. Acad. Ci. Madrid **52** (1958), 147–321; MR **21** #651]. The remaining two chapters deal, respectively, with total and partial orderings; much in them is closely connected with the author's papers that have appeared since 1942. There is a table of contents, but no index. {The three-page list of errata is incomplete.} *F. Bagemihl* (South Bend, Ind.)

2602a:

Cuesta Dutari, Norberto. Mathematics of order. III. Rev. Acad. Ci. Madrid **52** (1958), 609–770. (Spanish)

2602b:

Cuesta Dutari, Norberto. Mathematics of order. IV. Rev. Acad. Ci. Madrid **53** (1959), 33–190. (Spanish)

These are the last two chapters (without the table of contents and list of errata) of the author's book [reviewed above]. *F. Bagemihl* (South Bend, Ind.)

2603:

Kurepa, Georges. Sur la puissance des ensembles partiellement ordonnés. Sprawozd. Towarz. Nauk. Warszaw. Mat.-Fiz. **32** (1939), no. 1/3, 61–67. (Polish summary)

Let S be a partially ordered set, $\mathfrak{s}$ its power, $\mathfrak{w}$ the smallest cardinal number not less than the power of any one of its subsets that is well-ordered or inversely well-ordered, and $\mathfrak{i}$ the smallest cardinal number not less than the power of any one of its subsets E such that $a \in E$, $b \in E$ implies that a and b are incomparable. It is shown that $\mathfrak{s} \leq (2\mathfrak{i})^{\mathfrak{w}}$. This result, which was announced by the author [C. R. Acad. Sci. Paris **205** (1937), 1196–1198] and is proved also in his book [*Teorija skupova*, Školska Knjiga, Zagreb, 1951; MR **12**, 683; pp. 237–242], reduces to a theorem of Hausdorff when S is totally ordered (in which case $\mathfrak{i}=1$). {The author notes that the words "le dernier point ... point de F" following (10) on p. 63 should be replaced by "l'ensemble R_0B ou l'ensemble des premiers points de l'ensemble des points de B succédant à tout point de F suivant que F est vide ou non vide"; and that, on p. 67, l. 12, one should have $R_1E = R_0(E - R_0E)$, $R_iE = R_0(E - \sum_0^{i-1} R_kE)$.}

F. Bagemihl (South Bend, Ind.)

2604:

Erdös, P.; and Rado, R. A theorem on partial well-ordering of sets of vectors. J. London Math. Soc. **34** (1959), 222–224.

The authors prove the following theorem: For every ordinal $n < \omega^\omega$ and every "partially well-ordered" set S the set $V_S(<n)$ is partially well-ordered. This constitutes a proof, for a special case, of Rado's conjecture [Mathematika **1** (1954), 89–95; MR **16**, 576] that the corresponding statement holds independently of n. [For the terminology and notations see R. Rado, loc. cit.; cf. also Kurepa, C. R. Acad. Sci. Paris **205** (1937), 1196–1198; Acad. Serbe Sci. Publ. Inst. Math. **2** (1948), 26–36; MR **10**, 359.] *Đ. Kurepa* (Princeton, N.J.)

2605:

Climescu, Al. Sur un théorème de la théorie des treillis. Bul. Inst. Politehn. Iaşi (N.S.) **4** (**8**) (1958), 1–2. (Romanian. Russian and French summaries)

Let $P(M)$ be the lattice of all subsets of a set M. Let A be a subset of $P(M)$ which is a lattice under the ordering relation induced on A from $P(M)$ and which is complete under the greatest lower bound operation thus defined on A. H. Hermes [*Einführung in die Verbandstheorie*, Springer, Berlin-Göttingen-Heidelberg, 1955; MR **16**, 667; Satz 7.2, p. 35] gives a necessary and sufficient condition that a directed subset N of A have its set-theoretic union, as a collection of subsets which are members of $P(M)$, in A. Likewise, Hermes, in his Satz 7.4, p. 37ff., op. cit., gives necessary and sufficient conditions that a lattice V can be faithfully represented as the lattice of all subsets of some set M which are closed under a set of relations. The present author improves the sufficiency proof of this latter theorem by employing the former. *F. Haimo* (St. Louis, Mo.)

2606:

Rolf, Howard L. The free lattice generated by a set of chains. Pacific J. Math. **8** (1958), 585–595.

R. P. Dilworth [Trans. Amer. Math. Soc. **57** (1945), 123–154; MR **7**, 1] hat zu jeder teilweise geordneten Menge P den durch die Menge P erzeugten freien Verband $\mathrm{FL}(P)$ konstruiert. Der Verf. untersucht diesen freien Verband in dem Falle, daß P eine Kardinalsumme von endlich vielen endlichen Ketten ist, $P = n_1 + n_2 + \cdots + n_m$. Er beweist, daß $\mathrm{FL}(P)$ eine o-konvergente Folge mit voneinander verschiedenen Elementen enthält, wenn einer der folgenden Fälle eintritt: (1) $m \geq 3$; (2) $m \geq 2$ und es gibt ein i mit der Eigenschaft $n_i \geq 4$; (3) $m \geq 2$ und es gibt $i \neq j$ mit der Eigenschaft $n_i \geq 2$, $n_j \geq 2$. Ferner ist unter gewissen Voraussetzungen $\mathrm{FL}(1+1+1)$ ein Unterverband von $\mathrm{FL}(n_1+n_2)$ und umgekehrt. *M. Novotný* (Brno)

2607:

Hashimoto, Junji. Betweenness geometry. Osaka Math. J. **10** (1958), 147–158.

Unter einem B_1-Raum versteht der Verf. eine Menge mit einer ternären Relation ("zwischen"), welche einem gewissen Axiomensystem genügt. Die Teilmengen eines B_1-Raumes, welche auf eine gewisse Art in bezug auf die Relation "zwischen" abgeschlossen sind, heißen Unterräume; ein durch zwei verschiedene Elemente erzeugter Unterraum heißt Gerade. Zwei verschiedene Elemente liegen auf genau einer Geraden. Eine Gerade kann auf genau zweierlei Art so teilweise geordnet werden, daß die gegebene Relation "zwischen" mit der Relation "zwischen" identisch ist, welche aus der Ordnung abgeleitet ist. Ein Unterraum des B_1-Raumes, welcher durch drei nicht auf einer Geraden liegende Elemente erzeugt wird, heißt Ebene. Wenn zwei verschiedene Elemente einer Geraden in einer Ebene liegen, so liegt die Gerade in dieser Ebene. Wenn die Relation "zwischen" des B_1-Raumes einem weiteren Axiom genügt, so heißt der B_1-Raum ein B_2-Raum. Die beiden oben erwähnten Ordnungen auf den Geraden eines B_2-Raumes sind einfach. Zum Schluß werden solche B_2-Räume studiert, deren Ebenen dem Axiom von Pasch genügen. Irgend drei nicht auf ein und derselben Geraden liegende Elemente eines solchen

Raumes liegen auf genau einer Ebene. Eine Ebene wird durch eine Gerade ähnlich wie die Euklidische Ebene getrennt. *M. Novotný* (Brno)

2608:

★**Lesieur, L. Sur les treillis géométriques.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 11–19. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

The author re-examines a class of lattices which include the projective and affine geometries and their generalisations as well as the lattice of equivalence relations [see Dubreil-Jacotin, Lesieur and Croisot, *Leçons sur la théorie des treillis des structures algébriques ordonnées et des treillis géométriques*, Gauthier-Villars, Paris, 1953; MR 15, 279]. The axioms are formulated somewhat differently and it is shown that the continuity axiom may be weakened. *O. Ore* (New Haven, Conn.)

2609:

★**Permutti, Rodolfo. Spazi affini generalizzati e relative proprietà reticolari.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 120–121. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

The subspaces of a generalized affine space form a relatively complemented upper semi-modular lattice; the converse holds for length 3. The relationship is discussed briefly; no references are given, but most of this appeared in the author's paper in Ricerche Mat. **2** (1953), 192–203 [MR 15, 735]. Some related open questions are posed.

P. M. Whitman (Silver Spring, Md.)

2610:

Gemignani, Giuseppe. Su une teoria assiomatica della dipendenza. Rend. Mat. e Appl. (5) **17** (1958), 405–414.

Let S be a partially ordered set with 0, and N the subset of S such that if $n \in N$ then there is a maximal finite chain in S joining 0 to n. A quasi-order relation & on S is called a dependence relation if (1) $x \leq y$ implies $x \,\&\, y$; (2) whenever $T \subset S$ with $x \,\&\, y$ for all $x \in T$, and $\mathbf{V}_{x \in T} x$ exists, then $(\mathbf{V}_{x \in T} x) \,\&\, y$; (3) if $x, y, z \in S$ with y and z covering x and $z \,\&\, y$, z non-& x, then $y \,\&\, z$; (4) if $x \,\&\, y$ then $x \,\&\, n$ and $n \leq y$ for some $n \in N$. Any $x \in S$ is called degenerate if $x \,\&\, y$ for some $y < x$. Subset $T \subset S$ is called closed under & if $x \in T$ and $y \,\&\, x$ imply $y \in T$. Write $x \sim y$ if $x \,\&\, y$ and $y \,\&\, x$. If T is a closed subset of S, then a non-degenerate element $z \in T$ is called a base of T if $x \,\&\, z$ for all $x \in T$.

Theorem: If S is a discrete Boolean algebra in which each ascending chain has a join, then every $T \subset S$ has a base; moreover, if $y \in T$ is not degenerate then there exists a base z with $y \leq z$; if z and u are two bases of T, the set of atoms preceding z has the same cardinal number as the set for u. Some properties of & and $\sim$ are investigated, likewise for a set which is mapped homomorphically onto S. The last section gives some examples and discusses the mapping $x \to \mathbf{V}_{y \,\&\, x}\, y$ of S into itself.

P. M. Whitman (Silver Spring, Md.)

CLASSICAL ALGEBRA

2611:

Klamkin, M. S.; and Newman, D. J. On the number of distinct zeros of polynomials. Amer. Math. Monthly **66** (1959), 494–496.

The following theorems are proved: (1) Let $P(x)$ be a polynomial of degree n and $Q(x)$ a polynomial of degree m, $n > m \geq 0$. There exist $n-m+1$ distinct x's for which either $P(x)=0$ or $P(x)=Q(x)$. (2) Let $R(x)$ be a given polynomial of degree $c \geq 0$; then $P^a(x) - Q^b(x) = R(x)$ is impossible for polynomials (nontrivial) P and Q with complex coefficients if either a or b exceeds $2c$, $a > 1$, $b > 1$.

E. Frank (Chicago, Ill.)

THEORY OF NUMBERS

See also 2583, 2596, 2646, 2700, 2750, 2840, 2939.

2612:

★**Марчевский, М. Н. Теория чисел. Краткий курс.** [Marčevskiĭ, M. N. Theory of numbers. A short course.] Izdat. Har'kovsk. Gos. Univ. im A. M. Gorkiĭ, Kharkov, 1958. 144 pp. Paperbound: 4 rubles.

An elementary text with seven chapters: theory of division, questions connected with prime numbers, continued fractions, the most important number-theoretical functions, congruences, congruences with one unknown, exponential congruences and primitive roots and indices.

2613:

Eltermann, Heinz. Iterierte absolute Differenzen mit natürlichen Zahlen. J. Reine Angew. Math. **201** (1959), 71–77.

Let a row of n non-negative numbers $a_1, \cdots, a_n$ be given. Form a second row by putting $a_i' = |a_{i+1} - a_i|$ $(i = 1, 2, \cdots, n)$, where a_{n+1} means the number a_1. From the second row form the third by the same procedure, etc. The author investigates some properties of this algorithm for integers a_i and announces a second paper for the general case of real a_i. For the integer case the process becomes periodic after a finite number of steps; the author is especially interested in the question of when one is led to a row consisting exclusively of zeros and when not.

J. F. Koksma (Amsterdam)

2614:

Ingelstam, Lars. A remark on a problem on square roots. Nordisk Mat. Tidskr. **6** (1958), 153–154. (Swedish)

New proof of the fact: To every pair of integers n and p one can find a third, m, such that $(\sqrt{(n+1)} \pm \sqrt{n})^p = \sqrt{(m+1)} \pm \sqrt{m}$. *O. Ore* (New Haven, Conn.)

2615:

Tverberg, Helge. On two inequalities by S. Selberg. Math. Scand. **6** (1958), 308–310.

In this paper the author gives a short proof, using elementary methods, that $0 \leq \sigma_k(x) \leq k$, where

$$\sigma_k(x) = \sum_{\lambda=1}^{k} \left(x \left[\frac{\lambda}{x} \right] - (x+1) \left[\frac{\lambda}{x+1} \right] \right).$$

C. B. Haselgrove (Manchester)

2616:

Cohen, Eckford. Representations of even functions (mod r). II. Cauchy products. Duke Math. J. **26** (1959), 165–182.

The author continues the study of representations of even functions which was published in an earlier work [same J. **25** (1958), 401–421; MR **20** #5756]. Using these representations he gets different arithmetic identities as corollaries. The paper is mainly a collection of such corollaries. *H. Bergström* (Göteborg)

2617:

Sierpiński, W. Sur une décomposition des nombres premiers en deux classes. Collect. Math. **10** (1958), 81–83.

The following theorem is proved. The set A of all primes which are divisors of integers of form 2^r+1 contains all primes of the form $8n\pm3$ and infinitely many primes of the form $8n+1$. The set B of all primes which are divisors of integers of the form $2^{2s+1}-1$ contains all primes of the form $8n+7$ and some primes of the form $8n+1$. Every prime of the form $8n+1$ belongs either to A or to B. The question whether the set B contains infinitely many primes of the form $8n+1$ is raised, but remains open. {In a paper by the reviewer to be published in the Proc. Amer. Math. Soc., this question is answered in the affirmative.} *A. Brauer* (Chapel Hill, N.C.)

2618:

Pérez-Cacho, T. On some questions in the theory of numbers. Mem. Mat. Inst. "Jorge Juan", no. 20, 29 pp. (1958). (Spanish)

Same as article below.

2619:

Pérez-Cacho, L. On some questions in the theory of numbers. Rev. Mat. Hisp.-Amer. (4) **18** (1958), 10–27, 113–124. (Spanish)

The largest part of these two papers deals with the derivation of certain consequences of the assumption that the Fermat equation $x^n+y^n=z^n$ can be satisfied by positive integers x, y, z relatively prime in pairs (n an odd prime). Thus in the case where n is prime to x, y, z the author sets $x=aA$, where $(a, A)=1$ and $z-y=a^n$. He then proves that each prime factor of A is of the form Kn^2+1. A simple proof of Wieferich's theorem that $2^{n-1}\equiv1 \pmod{n^2}$ results from this and the similar facts about y and z. A further exploitation of the nature of the prime factors of x, y, z enables him to derive lower bounds for x and y. Thus for $n=1093$, y would have to be a number with at least 21860 digits. Similar results are derived in case n divides x; for $n=1093$ it then turns out that y would have to be a number with at least 14000 digits. Another typical result states that if n is of the form $6k-1$, then one of the numbers x, y, z must be divisible by 3.

In addition there are a few miscellaneous items mainly dealing with rather specialized arithmetic functions. For example, the author discusses the equation

$$n = \varphi(n)+\varphi(\varphi(n))+\cdots,$$

where $\varphi(n)$ is the Euler function.

H. W. Brinkmann (Swarthmore, Pa.)

2620:

Nagell, Trygve. Sur l'équation $x^5+y^5=z^5$. Ark. Mat. **3** (1958), 511–514.

The author gives a new proof of the non-existence of solutions of the Fermat equation

$$x^5+y^5 = z^5 \tag{*}$$

in non-zero integers of the quadratic field $K(\sqrt{5})$. The author's proof is independent of the general theory of Kummer, and is effected by considering, instead of (*), the more general equation

$$x^5-y^5 = E\cdot(\sqrt{5})^{5+\mu}z^5. \tag{**}$$

The non-existence of solutions of (**) is shown under the conditions that μ be a non-negative rational integer, E a unit of $K(\sqrt{5})$, and x, y, z non-zero integers of $K(\sqrt{5})$. The non-existence of solutions of (*) then follows from the fact, established in a lemma, that in any solution of (*) in non-zero integers of $K(\sqrt{5})$, at least one of x, y, z is divisible by $\sqrt{5}$. *A. Sklar* (Chicago, Ill.)

2621:

Nagell, Trygve. Sur une classe d'équations exponentielles. Ark. Mat. **3** (1958), 569–582.

The bulk of this paper is devoted to a case by case examination of the set of solutions in non-negative integers x, y, z of the exponential equations $a^x=b^y+c^z$, where a, b, c are distinct primes ≤7. There are nine cases to consider (obviously one of a, b, c must be 2). Five cases can be settled by elementary arguments based primarily on the set of values assumed by powers of one prime modulo a fixed power of another. Two, $(a, b, c)=(3, 5, 2)$ and $(3, 7, 2)$, require an argument based on the solutions of equations of the type $X^2-DY^2=C$. The cases $(2, 5, 3)$ and $(2, 5, 7)$ are dealt with by algebraic number theory using standard divisibility arguments involving ideals in quadratic fields and, for the equation $2^x=5^y+3$, the cubic field generated by $2^{1/3}$. All solution sets are finite, the maximum number of distinct solutions being 5 for $(a, b, c)=(2, 5, 3)$.

The detailed examination is preceded by some remarks on more general systems of exponential equations; in particular, a simple proof is given of the fact that

$$AM_1^{x_1}M_2^{x_2}\cdots M_m^{x_m}-BN_1^{y_1}N_2^{y_2}\cdots N_n^{y_n} = C$$

has only a finite number of non-negative integer solutions, $x_1, \cdots, x_m, y_1, \cdots, y_n$ where A, B, C, the M_i and the N_i are positive integers coprime in pairs.

J. D. Swift (Los Angeles, Calif.)

2622:

Malyšev, A. V. Theory of ternary quadratic forms. I. Arithmetic of hermitions. Vestnik Leningrad. Univ. **14** (1959), no. 7, 55–71. (Russian. English summary)

This is the first of a series of papers which will give in detail proofs of generalisations of results of Linnik [Izv. Akad. Nauk SSSR. Ser. Mat. **4** (1940), 363–402; MR **2**, 348] on the representation of large integers by definite ternary quadratic forms, and of the results of Linnik and the author about their uniform distribution [cf. Malyšev, same Vestnik **11** (1956), no. 19, 18–34; MR **20** #3124]. This paper deals with the arithmetic of Hermitions; that is, the generalized quaternions $x_0+x_1i_1+x_2i_2+x_3i_3$ with

norm-form $x_0^2+\Delta f(x_1, x_2, x_3)$, where f is a given ternary primitive positive-definite form of invariants $[\Delta, \Omega]$. An integral Hermition is one with integral coordinates $x_0, \cdots, x_3$.

In the first section (§§ 1–10) arithmetic and algebraic properties are developed from first principles without recourse to the general theory of algebras. A key result is that if M is a primitive integral Hermition (i.e. not divisible by a rational integer) whose norm is divisible by a rational integer r prime to $2\Delta\Omega$, and if R is a given primitive integral Hermition of norm r, then there exists an integral Hermition X_0 such that MX_0 is divisible by R on the right and such that $X_0\bar{R}$ is primitive (the bar denotes the anti-automorphism $\iota_j \to -\iota_j$ $(j=1, 2, 3)$). If, further, Norm $M=r$ and MX is divisible by R on the right, then $X \equiv uX_0 \pmod{R \text{ on right}}$ for some rational integer u. In the second section (§ 11) the author shows that results of his about the representation of numbers by quaternary forms [Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 337–364; cf. Tartakowsky, Izv. Akad. Nauk SSSR. Otd. Fiz.-Mat. Nauk **1929**, 111–122, 165–196] ensure the existence of Hermitions with special properties. In the third section (§ 12) the author generalizes to Hermitions the work of Venkov [ibid. **16** (1922), 205–246], which relates the theory of ordinary quaternions to that of binary quadratic forms. *J. W. S. Cassels* (Cambridge, England)

2623:

Walfisz, A. Über die Wirksamkeit einiger Abschätzungen trigonometrischer Summen. Acta Arith. **4** (1958), 108–180.

The author first restates various estimates for exponential sums, all obtained by methods due to Vinogradov, given by Tchudakov, Hua, Flett, Walfisz, Tatuzawa, and Vinogradov himself. The relations between these estimates are discussed, and they are applied (in combination with Weyl's estimate) to four well-known asymptotic problems. These are: (1) the estimation of $\zeta(1+it)$, (2) the error term in the prime number theorem, (3) the error term in the formula for the number of integer points in a large (rational) ellipsoid, (4) the error term for $\sum_{n \le x} \varphi(n)$. In (1), for example, the result in every case is of the form $O((\log t)^{3/4}(\log\log t)^{\alpha})$, with some particular constant α. Since results with exponents less then $\frac{3}{4}$ have now been announced [see the following review], the paper may soon be mainly of historical interest.

H. Davenport (Cambridge, England)

2624:

Vinogradov, I. M. A new estimate of the function $\zeta(1+it)$. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 161–164. (Russian)

This paper outlines a proof that

$$\zeta(1+it) = O((\log t)^{2/3}) \tag{1}$$

as $t \to \infty$, this being a better estimate than was previously known. It follows, by arguments that are now well known, that

$$\pi(x) = \operatorname{li} x + O\{x \exp(-C(\log x)^{3/5})\}, \tag{2}$$

where C is a positive constant. The result (1) is a consequence of the following estimate for the sum

$$S = \sum_{m=a}^{a_1} m^{-it}, \quad \text{where } a < a_1 \le 2a.$$

If $a \ge \exp(320000\, n^2)$ and $a^{n-1} \le t < a^n$, then $|S| < 1.04a^{1-\rho}$, where $\rho = (41792000\, n^2)^{-1}$. The special merit of this (and similar recent results) lies in the fact that the constants in it are absolute. The proof is based on an improved form of the author's mean-value theorem [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 109–130; MR **13**, 328; see also L. K. Hua, Quart. J. Math. **20** (1949), 48–61; MR **10**, 597]. This is stated without proof. [See also the review which follows.] *H. Davenport* (Cambridge, England)

2625:

Korobov, N. M. Weyl's estimates of sums and the distribution of primes. Dokl. Akad. Nauk SSSR **123** (1958), 28–31. (Russian)

The author outlines his own proof, obtained independently, of (1) and (2) of the preceding review. He proves (in reasonable detail) the following estimate for the exponential sum

$$S = \sum_{x=1}^{P} e(\alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{n+1} x^{n+1}),$$

where $e(t) = \exp 2\pi i t$. Suppose $\alpha_1, \cdots, \alpha_{n+1}$ are real, and

$$|\alpha_{n+1} - a/q| \le q^{-2}, \quad (a, q) = 1,$$

and $q = P^r$, where $n^{1/2}\log n < r < n - n^{1/2}\log n$. Then

$$|S| < CP^{1-\gamma/(n^2 \log n)},$$

where C and γ are positive absolute constants. The only essential result quoted is Vinogradov's mean value theorem [references in preceding review]. The above is theorem 1; in theorem 2 it is observed that the estimate can be improved, by the omission of $\log n$, if r is restricted to the narrower range $\varepsilon n < r < (1-\varepsilon)n$, for fixed $\varepsilon > 0$. Theorem 3 states a similar result for exponential sums in which several consecutive coefficients $\alpha_{s+2}, \cdots, \alpha_{2s}$ are rational. The proof is said to depend on an improved form of Vinogradov's mean-value theorem. It is from theorem 3 that the results (1) and (2) are stated to follow.

H. Davenport (Cambridge, England)

2626:

Gutmann, Hans. Anwendung Tauberscher Sätze und Lambertscher Reihen in der zahlentheoretischen Asymptotik. Verh. Naturf. Ges. Basel **69** (1959), 119–144.

As a generalization of the sequence of prime numbers the author considers a sequence B of natural numbers, $b_1 < b_2 < \cdots < b_k < \cdots$, which are prime to one another, and studies problems of asymptotic behavior of the number of numbers ($=b$-numbers) in B, the sum, the sum of reciprocal values, the sum of logarithms of numbers not divisible by the numbers of B, as well as the divisors and the sum of divisors which are divisible by the numbers of B. The idea to replace the prime numbers by the b-numbers is due to A. Beurling [Acta Math. **68** (1937), 255–291].

Throughout the paper convergence is assumed for $\sum_1^\infty b_\nu^{-1}$ and $\sum_1^\infty b_\nu^{-1} \cdot \log b_\nu$. Let the functions $f_n(x)$ be defined by the following relations ($|x| \ne 1$):

$$f_0(x) = \frac{x}{1-x}, \quad f_{n+1}(x) = f_n(x) - f_n(x^{b_{n+1}}), \quad n = 0, 1, 2, \cdots.$$

Then for $|x| < 1$ and for any n it is proved by mathematical induction that

$$f_n(x) = \sum_{\beta \in M^{(n)}} (-1)^{\omega^{(n)}(\beta)} \frac{x^\beta}{1-x^\beta},$$

where $M^{(n)}$ is the totality of the b^2-free numbers formed by the factors 1, b_1, b_2, $\cdots$, b_n, and $\omega^{(n)}(\nu)$ is the number of b-numbers by which ν is divisible. In connection with the functions $f_n(x)$ are defined two other similar functions with recurrent relations, and their limiting functions are studied when $n\to\infty$. The Lambert series $L(x)=\sum_1^\infty x^\nu/(1-x^\nu)$ plays an important role as may be seen from the form of the functions $f_n(x)$. It is deduced for $0\leqq x<1$ that

$$\sum_1^\infty \frac{\mu(\nu)x^\nu}{1-x^\nu} = x \quad \text{and} \quad \sum_1^\infty \frac{\mu(\nu)}{\nu}\log(1-x^\nu)^{-1} = x,$$

where $\mu(\nu)$ is the Möbius function. By applying Tauberian theorems of Hardy and Littlewood [G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949; MR **11**, 25; e.g. theorems 90, 91, 92, 96, 113] many interesting number-theoretic results are obtained. *S. Ikehara* (Tokyo)

2627:

David, Yosef. On a sequence generated by a sieving process. Riveon Lematematika **11** (1957), 26–31. (Hebrew. English summary)

An asymptotic expression is found for $e(n)$, the number of terms smaller than n in the sequence obtained from the positive integers by crossing out every kth term in the kth successive sequence, with $k=2, 3, \cdots$. The same sieving process is considered in an article by Erdös and Jabotinsky [reviewed below]. *From the author's summary*

2628:

Erdös, Paul; and Jabotinsky, Eri. On sequences of integers generated by a sieving process. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 115–128.

The authors' algorithms depend on an initial integer λ and on an auxiliary sequence B of integers b_k ($k=1, 2, \cdots$) with $b_k\geqq 2$. A family of intermediary sequences $A^{(i)}$ ($i=1, 2, \cdots$) is formed. $A^{(i)}$ consists of the integers $a_k^{(i)}$ ($k=1, 2, \cdots$). The sequence $A^{(1)}$ is defined by: $a_k^{(1)}=\lambda+k$. We obtain $A^{(i+1)}$ from $A^{(i)}$ by striking out all the terms of the form $a^{(i)}_{1+mb_i}$ ($m=0, 1, \cdots$) and then renaming the sequence of remaining terms $a_1^{(i+1)}, a_2^{(i+1)}, a_3^{(i+1)}, \cdots$. Finally, the sequence A consisting of integers a_k ($k=1, 2, \cdots$) is defined by $a_k=a_1^{(k)}$.

As an example, take $b_k=k+1$ and $\lambda=0$. Then the first nine a's are 1, 2, 4, 6, 10, 12, 18, 22, 30. A second example is $b_k=a_k$ with $\lambda=1$. Then the first ten a's are 2, 3, 5, 7, 11, 13, 17, 23, 25, 29.

We quote only two of the authors' results: I. For $b_k=k+1$ we have

$$a_k = \frac{k^2}{\pi}+O(k^{4/3});$$

II. For $b_k=a_k$ we have (γ Euler's constant)

$$(1)\qquad a_k = k\log k+\tfrac{1}{2}k(\log\log k)^2 + (2-\gamma)k\log\log k+o(k\log\log k).$$

The dominant term here was also obtained independently by D. Hawkins and W. E. Briggs and, following these authors, the reviewer noted the second term in (1) [Hawkins learnt of these algorithms from S. Ulam. See the paper by V. Gardiner, R. Lazarus, N. Metropolis and S. Ulam in Math. Mag. **29** (1956), 117–122; MR **17**, 711]. *S. Chowla* (Boulder, Colo.)

2629:

Hawkins, D.; and Briggs, W. E. The lucky number theorem. Math. Mag. **31** (1957/58), 81–84.

The authors define a sequence of integers $a_1<a_2<\cdots$ by a sieve process as follows: Let S_2 be the sequence 2, 3, 5, 7, $\cdots$. Assume S_n has been already defined and that the elements of S_n are $t_{n,m}$, $1\leqq m<\infty$. S_{n+1} is obtained from S_n by crossing out every $t_{n,n}$-th element of S_n. {It seems to the reviewer that the authors' definition is inaccurate in going from S_1 to S_2.}

The authors prove by an elementary Tauberian argument that $a_k/(k\log k)\to 1$. This result is analogous to the prime number theorem. {Substantially the same result has been proved independently by Jabotinsky and the reviewer [preceding review].} *P. Erdös* (Birmingham)

2630:

Hawkins, D.; and Briggs, W. E. The lucky number theorem. Math. Mag. **31** (1957/58), 277–280.

Identical with the article reviewed above.

2631:

Erdös, P.; and Scherk, P. On a question of additive number theory. Acta Arith. **5** (1958), 45–55 (1959).

Let $\mathscr{A}_1=\{a_i^{(1)}\}$, $\mathscr{A}_2=\{a_i^{(2)}\}$, $\cdots$ denote sequences of positive integers, and let $A_1(n)$ denote the number of elements of $\mathscr{A}_1$ up to and including n. Addition of sequences is effected in the manner of Schnirelmann, e.g., $\mathscr{A}_1+\mathscr{A}_2$ is the sequence consisting of all the distinct positive integers that are representable as $a_i^{(1)}$ or $a_i^{(2)}$ or $a_i^{(1)}+a_j^{(2)}$. If n does not belong to $\mathscr{A}_1+\cdots+\mathscr{A}_k$ it is almost immediate that

$$A_1(n)+\cdots+A_k(n)\leqq \tfrac{1}{2}k(n-1).$$

The authors consider the value of the sum on the left when n is the least positive integer which does not belong to $\mathscr{A}_1+\cdots+\mathscr{A}_k$. Denoting the upper bound of the sum, for all sequences satisfying the condition, by $f_k(n)$, they prove the existence of positive numbers α_k and γ_k such that

$$\tfrac{1}{2}kn-\alpha_k n^{1-1/k} < f_k(n) < \tfrac{1}{2}kn-\gamma_k n^{1-1/k}.$$

The proofs are elementary but difficult. *H. Davenport* (Cambridge, England)

2632:

Mineev, M. P. The Tarry problem for rapidly increasing functions. Mat. Sb. N.S. **46 (88)** (1958), 451–454. (Russian)

Theorem: Let $g_0=1$ and let $g_i\geqq 2$ ($i\geqq 1$) be integers. Put

$$F(x) = g_0g_1\cdots g_x.$$

Then for fixed n the number of solutions in positive integers of the diophantine equation $F(x_1)+\cdots+F(x_n)=F(y_1)+\cdots+F(y_n)$ with $0\leqq x_j<p$, $0\leqq y_i<p$, is $n!p^n+O(p^{n-1})$.

The author shows, in fact, quite easily, that the number of solutions for which the x_j are not a permutation of the y_i is $O(p^{n-1})$. The theorem generalizes a result of Postnikov [Festschrift anlässlich des 250. Geburtstages Leonhard Eulers, Berlin, 1957] for the case when the g_i are all equal,

and also an estimate of Loo-keng Hua [Acta Sci. Sinica **1** (1952), 1–76; MR **16**, 337] for the ordinary Tarry problem $F(x)=x^k$. *J. W. S. Cassels* (Cambridge, England)

2633:

Schinzel, André. Démonstration d'une conséquence de l'hypothèse de Goldbach. Compositio Math. **14**, 74–76 (1959).

The author proves that: (1) for any positive integers k, m there exist arbitrarily large primes p, q such that $2k \equiv p+q \pmod m$, and (2) $2k \equiv p-q \pmod m$; (3) for any positive integer m there is an arbitrarily large prime q such that $2^p-1 \equiv q \pmod m$ if prime p is large enough; (4) for any positive integer m and a sufficiently large positive integer n, $2^{2^n}+1 \equiv p \pmod m$ for infinitely many primes p. (1), (3) and (4) have been proved by Sierpiński under the assumption of the Goldbach conjecture. The present proofs are elementary consequences of Dirichlet's theorem on primes in arithmetic progressions and a lemma proved by the author: let $f(x)$ be a polynomial over rational integers, let $d \mid f(x)$ for all rational integers x and $d \geqq 1$ imply $d=1$, and let m be a positive integer; then $(f(x_0), m)=1$ for a rational integer x_0.

Z. A. Melzak (Montreal, P.Q.)

2634:

Narayana, T. V.; and Fulton, G. E. A note on the compositions of an integer. Canad. Math. Bull. **1** (1958), 169–173.

A composition of an integer is a partition in which the order of the parts is significant (e.g., 1, 2, 3 is not the same as 1, 3, 2). The authors show that the compositions of n into r parts form a distributive lattice with $t_1, \cdots, t_r$ including $t_1', \cdots, t_r'$ if $t_1+\cdots+t_i \geqq t_1'+\cdots+t_i'$ for $i=1, 2, \cdots, r$. They define a natural anti-isomorphism between this lattice and that of the compositions of n into $n-r+1$ parts, and show that it yields a one-one correspondence between the sets of compositions of n with elements $\leqq 2$, and of $n+2$ with elements $\geqq 2$.

K. Goldberg (Washington, D.C.)

2635: MANZURHUSSAIN, S.

Atkin, A. O. L.; and Hussain, S. M. Some properties of partitions. II. Trans. Amer. Math. Soc. **89** (1958), 184–200.

This paper is a sequel to one by Atkin and Swinnerton-Dyer [Proc. London Math. Soc. (3) **4** (1954), 84–106; MR **15**, 685]. Its main object is to give a complete account of Dyson's rank function for the case $q=11$. The rank of a partition is defined as the largest part minus the number of parts. For each q, the partitions of n are classified according to their rank mod q, and the number of partitions of n with rank congruent to m mod q is denoted by $N(m, q, n)$. The authors obtain identities for the series $r_{5c}(d)=r_5(d)-r_c(d)$, where $r_5(d)=\sum N(b, q, qn+d)y^n$, in terms of simpler series and products. The methods, although closely related to the theory of modular functions, are essentially elementary and involve lengthy algebraic manipulations. Among the results obtained is an identity for $\sum p(11n+6)y^n$ which proves the Ramanujan congruence $p(11n+6) \equiv 0 \pmod{11}$. {As pointed out by Atkin in a letter to the reviewer, there is some overlap with a paper by the reviewer [Tôhoku. Math. J. (2) **8** (1956), 149–164; MR **19**, 392]. The authors' functions λ, μ are related to the reviewer's α, β by $\beta=-\mu+3\lambda-23$, $\alpha=-\lambda-13$, and the authors' equations 8.13, 11.7, 11.9 are, respectively, the reviewer's equations 3.24, 3.22, 3.25. This overlapping is fortunate, since it brought to light an arithmetical error in the reviewer's paper: the last term written in his equation 3.20 for α should be $7x^4$, not $4x^4$; the right side of equation 3.23 should be α^5, not $\alpha^5+9(11^2+\alpha)^2$; equation 3.24 should read $\alpha^3+38\alpha^2+3.11^2\alpha=\beta^2+6\alpha\beta+11^2\beta$.} *N. J. Fine* (Princeton, N.J.)

2636:

Gupta, Hansraj. Partitions of j-partite numbers. Res. Bull. Panjab Univ. no. 146, 119–121 (1958).

Another proof of the author's result [J. London Math. Soc. **33** (1958), 403–405; MR **20** #7007] on the partitions of j-partite numbers. *J. Lehner* (East Lansing, Mich.)

2637:

Rényi, A. On Cantor's products. Colloq. Math. **6** (1958), 135–139.

As a Cantor product expansion of a real number $x>1$ the author defines the infinite product $\prod (1+1/q_\nu)$, where the q_ν are determined by the rule: Let q_1 be the least positive integer for which $1+1/q_1 \leqq x$; if $q_1, \cdots, q_{n-1}$ have been chosen, let q_n be the least positive integer such that $\prod_{\nu=1}^{n} (1+1/q_\nu) \leqq x$. Clearly there is a flaw in this definition, since according to it certain rational numbers (and as a matter of fact all rational numbers) have finite expansions. This leads to a certain amount of confusion in the early part of the paper. A more standard definition is one in which strict inequality is assumed in the relations defining the q_ν. Then every number $x>1$ has an infinite expansion, and a rational number is characterized by the fact that from some n_0 on $q_{n+1}=q_n^2$. The Cantor product expansion of a number $x>1$ is unique, and thus, for every n, $q_n=q_n(x)$. It is also known that $q_{n+1}(x) \geqq q_n^2(x)$.

For other expansions of real numbers, such as decimal expansions, regular continued fraction expansions, etc., ergodic properties of the coefficients of the expansion have been extensively investigated. This is the first time that ergodic results are obtained for Cantor products. Typical of the results stated (proofs are only sketched) in this article are the following: For almost all $1<x \leqq 2$

$$\lim_{n\to\infty} (q_{n+1}(x))^{1/2^n} = l(x),$$

where $2<l(x)<\infty$, and

$$\lim_{n\to\infty} [q_{n+1}(x)/q_1(x)q_2(x)\cdots q_n(x)]^{1/n} = e.$$

W. J. Thron (Boulder, Colo.)

2638:

Turán, Paul. Eine neue Methode in der Analysis und deren Anwendungen. Advancement in Math. **2** (1956), 312–565. (Chinese)

A Chinese version of the original German [Akadémiai Kiado, Budapest, 1953; MR **15**, 688].

2639:

Obrechkoff, Nikola. Sur l'approximation diophantienne des formes linéaires. Ark. Mat. **3** (1958), 537–542.

The author uses Dirichlet's Schubfachschluss to prove the following result, and deduces some corollaries. Let $n_1, \cdots, n_p, m_1, \cdots, m_p$ be positive integers. Let a_{ij} $(1 \leqq i \leqq p,\ 1 \leqq j \leqq n_p)$ be real. Then there exist integers x_{ij} and y such that (1) for each i the numbers x_{ij} $(1 \leqq j \leqq n_p)$ are either all non-negative or all non-positive, (2) $|x_{ij}| \leqq m_i$,

$$\text{(3)} \qquad \left| \sum_{i=1}^{p} \sum_{j=1}^{n_p} a_{ij} x_{ij} - y \right| \leqq \prod_{i=1}^{p} (n_i m_i + 1)^{-1}.$$

{In the array (3) on p. 538, m_{r-1} should be $m_r - 1$, and similarly for m_{r-2}.} *H. Davenport* (Cambridge, England)

2640:

Fel'dman, N. I. Transcendence of numbers of certain classes. Uspehi. Mat. Nauk **14** (1959), no. 1 (85), 237–244. (Russian)

The author proves the following theorem and uses it to prove the transcendency of decimal fractions and series: "If α and $k > 1$ are real constants, and if the inequality $|\alpha - p/q| < q^{-k}$ has infinitely many solutions in integers p, q such that $(p, q) = 1$, $q = q'q''$, where q' and q'' are positive integers such that $\lim (\log q'/\log q) = 0$ and q'' has only bounded prime factors, then α is transcendental." A slightly more general result was already published by D. Ridout [Mathematika **4** (1957), 125–131; MR **20** #32]. *K. Mahler* (Manchester)

2641:

Yakabe, Iwao. Note on arithmetic properties of Kummer's function. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 49–52.

Let $J_{\kappa,\lambda}(x) = \exp(-x/2) \cdot {}_1F_1(\kappa; \lambda; x)$, where ${}_1F_1(\kappa; \lambda; x)$ denotes the confluent hypergeometric function

$$1 + \kappa x/(\lambda \cdot 1!) + \kappa(\kappa+1)x^2/(\lambda(\lambda+1) \cdot 2!) + \cdots.$$

Let κ and λ be rational numbers, such that κ and $\kappa - \lambda$ are not integers. The author applies Siegel's method of E-functions in order to prove that for any algebraic $\alpha \neq 0$ the numbers $J_{\kappa,\lambda}(\alpha)$ and $J_{\kappa,\lambda}'(\alpha)$ are algebraically independent over the field of algebraic numbers [cf. e.g., C. L. Siegel, *Transcendental numbers*, Princeton Univ. Press, 1949; MR **11**, 330; Chapter II]. It follows, e.g., that at least one of the two values ${}_1F_1(\kappa; \lambda; \alpha)$, ${}_1F_1'(\kappa; \lambda; \alpha)$ is transcendental. There is no reference to the results of Šidlovskiĭ on E-functions [cf. A. B. Šidlovskiĭ, Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 53–66; MR **21** #1295; and earlier publications]. *J. Popken* (Amsterdam)

2642:

Ridout, D. Indefinite quadratic forms. Mathematika **5** (1958), 122–124.

Let Q be an indefinite quadratic form in n variables with real coefficients. Let r be the index of the form (the number of positive terms in the diagonal form after a real transformation). The author proves that if $\min(r, n-r) = 5$ and $n \geqq 21$, then, for any positive ε, the inequality $|Q| < \varepsilon$ is solvable in integers not all zero. This fills a gap between results of Birch and Davenport (for $\min(r, n-r) \leqq 4$) and those of Davenport and the author. *B. W. Jones* (Mayaguez, P.R.)

2643:

Kneser, Martin. Kleine Lösungen der diophantischen Gleichung $ax^2 + by^2 = cz^2$. Abh. Math. Sem. Univ. Hamburg **23** (1959), 163–173.

Let a, b, c be square-free positive integers which are relatively prime in pairs. Legendre found the necessary and sufficient conditions for the equation $ax^2 + by^2 = cz^2$ to have a solution in integers x, y, z, not all 0. Holzer [Canad. J. Math. **2** (1950), 238–244; MR **12**, 11] proved that, if the conditions are satisfied, there is a solution with $0 < z \leqq (ab)^{1/2}$. The present paper contains a somewhat more elementary proof of a result which includes this and sometimes asserts more. For each positive integer n there is defined, by consideration of plane lattices and their sublattices, a number $k(n)$, and it is proved that the equation is soluble with $0 < z \leqq k(n)(ab)^{1/2}$, where n is any divisor of the least common multiple of 2 and abc. Since $n = 2$ is admissible, and $k(2) = 1$, this includes Holzer's result. A process is described for determining $k(n)$, and a selection of numerical values is given.

H. Davenport (Cambridge, England)

2644:

Rédei, Ladislaus. Natürliche Basen des Kreisteilungskörpers. I. Abh. Math. Sem. Univ. Hamburg **23** (1959), 180–200.

Let G_n denote the group of all nth roots of unity, and let K_n be the number field generated by G_n. A subset of $\varphi(n)$ elements of G_n is called a natural base for K_n if all other elements of G_n can be expressed as linear combinations of the elements of this base, with coefficients 0 or ± 1 only. The construction of such natural bases is easily reduced to the case that n is squarefree, and for that case the author shows that the set of all primitive nth roots of unity is a natural base (he conjectures that all other natural bases are obtained from this special one just by rotation; the reviewer remarks that there is already a counterexample if $n = 6$, where $1, e^{2\pi i/6}$ form a natural base).

In connection with these questions the author studies balanced sets (Gleichgewichtsmengen) of roots of unity. A subset of G_n is called balanced if its sum is zero. G_n itself is trivially balanced $(n > 1)$, but non-trivial examples can be obtained as follows: Take $n = n_1 n_2 \cdots n_r$ $(r \geqq 3,\ r$ odd, $(n_i, n_j) = 1$ if $i \neq j)$; take $M_i \subset G_{n_i}$, M_i balanced, and split each M_i into non-empty parts A_i, B_i. Then $A_1 \cdots A_r + B_1 \cdots B_r$ is a balanced set. It is shown that a set $M \subset G_n$ is balanced if and only if there is a collection of trivial balanced sets $A_1, \cdots, A_r$ such that $A_1, \cdots, A_r, M$ are mutually disjoint and such that $A_1 + \cdots + A_r + M$ is a union of disjoint trivial balanced sets.

Next it is shown that the problem of the construction of all balanced sets can be reduced to the case that n is squarefree, and for that case the author gives a simple criterion for a set to be balanced. To this end he introduces a notion called "divisibility" and denoted by "$\omega | \nu$". If $n = p_1 \cdots p_r$, and if $\nu \in G_n$, then $\nu = \rho_1 \cdots \rho_r$, where $\rho_i \in G_{p_i}$, and the ρ_i are uniquely determined. Now every product of a subset of these ρ_i's is called a "divisor" of ν. Furthermore, denoting the order of an element $\omega \in G_n$ by $o(\omega)$, the author shows that a set $M \subset G_n$ is balanced if and only if $\sum_{\omega | \nu} \mu(o(\omega))\chi(\omega) = 0$ for all primitive $\nu \in G_n$ (μ is the Möbius function, and χ is the characteristic function of M).

If n is a prime or a product of two different primes, there

are only trivial balanced sets. For the case that n is the product of three different primes the author gives a general procedure for finding all balanced sets. The construction has the nature of the $A_1 \cdots A_r + B_1 \cdots B_r$ explained above, but is somewhat more complicated.

N. G. de Bruijn (Amsterdam)

FIELDS

See also 2620, 2644, 2665, 3015.

2645:

Vandiver, H. S. On conditional fractional equations in a field, involving one unknown. Math. Mag. **32** (1958/59), 173–176.

Let $F(x) = \sum_{j=1}^{j=k} N_j(x)/f_j(x)$ be a fractional equation with coefficients belonging to some field F, and set $F(x) = R(x)/M(x)$ where $M(x)$ is the L.C.M. (least common multiple) of the $f_j(x)$ $(j = 1, 2, \cdots, k)$, or any multiple of the L.C.M. but containing only irreducible factors of the L.C.M. Further, denote by $D(x)$ the G.C.D. (greatest common divisor) of $R(x)$ and $M(x)$ and write $R(x) = D(x)T(x)$. Then $T(x) = 0$ and $F(x) = 0$ are equivalent equations in F.

J. W. Andrushkiw (Newark, N.J.)

2646:

Furuta, Yoshiomi. A reciprocity law of the power residue symbol. J. Math. Soc. Japan **10** (1958), 46–54.

Soit k un corps de degré fini de nombres algébriques contenant une racine primitive l-ième ζ de l'unité (mais pas forcément les racines l^n-ième primitives de l'unité pour $n > 1$). L'auteur définit, pour les idéaux premiers $\mathfrak{p}$ tels que $N\mathfrak{p} \equiv 1 \pmod{l^n}$ (où $N\mathfrak{p}$ désigne la norme absolue de $\mathfrak{p}$) et pour certains $\alpha \in k$, un symbole $\left[\frac{\alpha}{\mathfrak{p}}\right]_n$ de restes des puissances l^n-ièmes (mod $\mathfrak{p}$) (dont celui de S. Kuroda [même J. **3** (1951), 148–156; MR **13**, 442] est un cas particulier pour $l = 2$, $n = 2$), et si, pour toutes les unités totalement positives ε de k, $\left[\frac{\varepsilon}{\mathfrak{p}}\right]_n = 1$, également le symbole $\left[\frac{(\alpha)}{\mathfrak{p}}\right]_n$. Il prouve une loi de réciprocité pour ce symbole quand k est rationnel et $l = 2$.

$\left[\frac{\alpha}{\mathfrak{p}}\right]_n$ est défini, par induction sur n, seulement pour les $\alpha \in k$ tels que $\left[\frac{\alpha}{\mathfrak{p}}\right]_{n-1} = 1$. Pour un tel α, $\alpha^{(N\mathfrak{p}-1):l^n}$ est congruent (mod $\mathfrak{p}$) à une racine l-ième de l'unité, et $\left[\frac{\alpha}{\mathfrak{p}}\right]_n$ est, par définition, cette racine de l'unité.

La loi de réciprocité mentionnée est trop compliquée pour être formulée ici. Elle exprime $\left[\frac{p}{q}\right]_n \left[\frac{q}{p}\right]_n$ à l'aide des symboles des puissances 2^{n-1}-ièmes, mais dont l'idéal et l'argument dépendent de p et de q.

{Remarque du référent: le lemme 5 de l'auteur est exact, mais sa démonstration dans le travail ne l'est pas.}

M. Krasner (Paris)

2647:

Cohn, Richard M. An improved result concerning singular manifolds of difference polynomials. Canad. J. Math. **11** (1959), 222–234.

In a preceding paper [Ann. of Math. (2) **53** (1951), 445–463; MR **13**, 103] the author found for difference polynomials a necessary condition that a given prime reflexive difference polynomial ideal (in one indeterminate) of effective order n be a singular component of a given difference polynomial of effective order $n + k$; this condition is analogous to but more complicated than the condition in J. F. Ritt's "low power theorem" for differential polynomials, which is sufficient as well as necessary. For difference polynomials the author proved [loc. cit. supra, and ibid. **57** (1953), 524–530; MR **14**, 721] that if $k = 1$ the condition is never satisfied, if $k = 2$, $n = 0$ the condition is sufficient, and if $k > 2$ the condition is not sufficient and no similar condition can be. In the present paper he modifies the method of his second cited paper to show that if $k = 2$ the condition is sufficient regardless of the value of n.

E. R. Kolchin (New York, N.Y.)

ALGEBRAIC GEOMETRY

See also 2671, 2750, 2993, 2994.

2648:

Rosina, Bellino Antonio. Sui multilateri sghembi connessi in relazione alla classificazione delle curve algebriche sghembe. Boll. Un. Mat. Ital. (3) **13** (1958), 525–530.

Dans cette note l'a. donne un résumé de ses études sur la classification des courbes gauches algébriques et des multilatères qui les charactérisent [voir à cet égard Ann. Univ. Ferrara, Sez. VII (N.S.) **6** (1956/57), 41–72, 73–120; MR **20** #870].

M. Piazzolla-Beloch (Ferrara)

2649:

Spampinato, Nicolò. La V_5 di S_3 determinata dalle coniche osculatrici di una curva algebrica piana prolungata nel campo tripotenziale. II. Ricerca, Napoli (2) **7** (1956), luglio-dicembre, 19–28.

Continued from same Ricerca **7** (1956), gennaio-giugno, 43–58 [MR **18**, 415].

2650:

Godeaux, Lucien. Sur certaines surfaces algébriques contenant des involutions cycliques n'admettant que des points unis de première espèce. Arch. Math. **9** (1958), 155–160.

A surface F in S_4 of order p^2 (p being any odd prime) is defined by equations

$$\text{(i)} \qquad \varphi_i(x_0, x_1) + \psi_i(x_2, x_3, x_4) = 0 \quad (i = 1, 2)$$

where φ_i, ψ_i are p-ic forms, φ_1, φ_2 have no common factor, and the plane curves $\psi_1 = 0$, $\psi_2 = 0$ have p^2 distinct intersections. This has on it a cyclic involution generated by the transformation

$$\text{(ii)} \qquad x_0' : x_1' : x_2' : x_3' : x_4' = x_0 : x_1 : \varepsilon x_2 : \varepsilon x_3 : \varepsilon x_4,$$

where ε is a primitive pth root of unity. A first model F^1 of the involution is obtained by taking

$$X_i = x_0^{p-i}x_1^i, \quad Y_{ij} = x_2^i x_3^j x_4^{p-i-j}$$

as coordinates in S_r, $r=\frac{1}{2}(p^2+5p+2)$; the lines of S_4 meeting the plane $x_0=x_1=0$ and the line $x_2=x_3=x_4=0$ are mapped on the lines of S_r meeting a normal rational p-ic curve in $Y_{ij}=0$ and a generalised Veronese surface of order p^2 (projective model of all p-ic curves in a plane) in $X_i=0$; each point of the M_4 generated by these ∞^3 lines corresponds to a set of the involution generated by (ii) in S_4, and the equations (i) give two hyperplanes in S_r whose intersection with M_4 is the surface F^1. The only united points of the involution on F are its p^2 intersections with $x_0=x_1=0$, and these are of the first kind, i.e. every point of the first neighbourhood is united. Their images on F^1 are p-ple points at which the tangent cone is rational and normal.

F has genus $p_g=p_a=p^2(7p^2-30p+35)/12-1$, and linear genus $p^{(1)}=p^2(2p-5)^2+1$. F^1 has genus

$$p_g{}^1 = p_a{}^1 = \tfrac{1}{2}(p-1)^2(p-2)$$

and linear genus

$$\pi = p(p-3)(3p-7)+1.$$

The canonical system on F is traced by all hypersurfaces of order $2p-5$; thus for $p=3$, F is its own canonical model. The image of the canonical system on F^1 is traced on F by the subsystem of these hypersurfaces of the form

$$\text{(iii)} \qquad \sum_{i=0}^{p-2} x_0^{p-2-i}x_1^i\alpha_i(x_2, x_3, x_4) = 0,$$

where $\alpha_0, \cdots, \alpha_{p-2}$ are forms of degree $p-3$. For $p=3$ the canonical system on F^1 is a pencil of elliptic curves. For $p\geqq 5$ a canonical model of F^1 is constructed by taking the $p_a{}^1$ separate monomials in (iii) as coordinates in $S_{p_a{}^1-1}$, when the lines of S_4 meeting $x_0=x_1=0$ and $x_2=x_3=x_4=0$ are mapped on the points of an M_3 which is the Segre product of a normal rational curve of order $p-2$ and a generalised Veronese surface of order $(p-3)^2$. The canonical model of F^1 is traced on this by all hypersurfaces of the form

$$\varphi_1\psi_2-\varphi_2\psi_1 = 0$$

where the variables in φ_i, ψ_i are replaced by coordinates in $S_{p_a{}^1-1}$ corresponding to monomials in $(x_0, x_1, x_2, x_3, x_4)$ in which only one of these takes the values (x_0, x_1) or (x_2, x_3, x_4), the rest remaining fixed. *P. Du Val* (London)

2651:

Engel, Wolfgang. Ganze Cremona-Transformationen von Primzahlgrad in der Ebene. Math. Ann. **136** (1958), 319–325.

It is shown that the equations $x'=g(x, y)$, $y'=h(x, y)$ define an integral cremona transformation of prime order p if and only if

$$g(x, y) = g_0+g_{10}x+g_{01}y + \beta\sum_{i=2}^{p}\alpha_i[(\alpha g_{10}-\beta h_{10})x+(\alpha g_{01}-\beta h_{01})y]^i,$$

$$h(x, y) = h_0+h_{10}x+h_{01}y + \alpha\sum_{i=2}^{p}\alpha_i[(\alpha g_{10}-\beta h_{10})x+(\alpha g_{01}-\beta h_{01})y]^i,$$

with $g_{10}h_{01}-g_{01}h_{10}\neq 0$. *G. B. Huff* (Athens, Ga.)

2652:

Manin, Yu. I. Algebraic curves over fields with differentiation. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 737–756. (Russian)

Le but de ce travail est de construire un homomorphisme du groupe des classes de diviseurs de degré zéro d'une courbe algébrique définie sur un corps de caractéristique zéro, dans le groupe additif d'un espace vectoriel de dimension finie sur le corps de base. Quelques propriétés du noyau de cet homomorphisme sont décrites. Comme application des résultats obtenus on établit un critère de dépendence linéaire de points sur une courbe elliptique. (L'auteur estime que ce critère peut être utile en connection avec les travaux de Néron sur le rang des courbes algébriques.) *E. Lluis* (Mexico, D.F.)

2653:

Burau, Werner. Untersuchungen zur Geometrie der projektiven Formen und Invarianten Theorie. I. Invariante Bildungen einer Einzelform. Collect. Math. **10** (1958), 21–43.

Starting with a projective space of n dimensions X_n, the author introduces the corresponding spaces P_m, $m=\binom{n+k}{n}-1$, of the forms of degree k in X_n, and finally the spaces S_r corresponding in the same way to the hypersurfaces of degree g in P_m. After having studied the relations induced by grassmannian manifolds and veronesian manifolds corresponding to degenerated hypersurfaces in X_n and P_m, the decompositions of the spaces S_r are studied, which provide geometrical interpretations of relations between projective invariants and comitants. The general scheme is worked out for an $(n+1)$-ary form of degree k and is given in detail for the binary quadratic, cubic and biquadratic form as well as for the ternary quadratic and cubic form. For these simple cases the direct methods of the theory of invariants do put into evidence the author's results on syzygies, but one should agree with the author that nevertheless it is of importance to have an approach to the problems from this side too.

E. M. Bruins (Amsterdam)

2654:

Washnitzer, G. Geometric syzygies. Amer. J. Math. **81** (1959), 171–248.

L'auteur donne diverses interprétations géométriques de suites exactes de faisceaux algébriques cohérents et donne, par généralisation de la classique théorie des syzygies, une caractérisation axiomatique du genre arithmétique d'une variété projective non singulière. Les résolutions au sens géométrique, ici introduites, sont définies à partir d'une suite exacte de faisceaux, pris sur une variété X, projective, non singulière et irréductible, de dimension r,

$$F_i \to \cdots \to F_s \xrightarrow{\psi_s} F_{s-1} \to \cdots \to F_0 \xrightarrow{\psi_0} Q \to 0,$$

les $F_0, \cdots, F_i$ étant localement libres sur X, Q une extension à X d'un faisceau localement libre sur V, sous-variété propre fixée de dimension d de X. On soit que $\operatorname{Ker}\psi_s[F_s\to F_{s-1}]$ est localement libre sur X si $s\geqq r-d-1$, et qu'il existe alors des suites exactes du type

$$0 \to F_{r-d} \xrightarrow{\psi_{r-d}} \cdots \to F_s \xrightarrow{\psi_s} F_{s-1} \to \cdots \xrightarrow{\psi_1} Q \to 0$$

avec $F_0, \cdots, F_{r-d}$ localement libres (résolution de Q). On remplace alors chaque faisceau E par une certaine variété projective Y_E (dual projective bundle) et chaque homomorphisme par une certaine transformation rationnelle $B(\psi)$. L'étude est limitée au cas où les graphes des diverses transformations rationnelles introduites sont sans point singulier, au moins sur certaines sous-variétés, ce qui est suffisant pour traiter le cas où V est de codimension 1 sur X, ou encore lorsque la dimension de Q sur V est la même que celle de E sur V.

Comme application on définit axiomatiquement un genre arithmétique $A(X)$ pour tout modèle projectif non singulier X qui est d'ailleurs égale à la caractéristique d'Euler-Poincaré du faisceau Ω_X des anneaux locaux de X. On peut supposer la caractéristique du corps de base quelconque. Sous le nom de "Fiber Law" l'auteur prouve que $A(X)=A(Y_E)$, Y_E désignant la variété projective associée à un faisceau localement libre E, attaché à un modèle projectif non singulier X. On annonce pour une publication ultérieure l'expression de $A(X)$ au moyen d'un genre de Todd. *J. Guérindon* (Rennes)

2655:

Nishimura, Toshio. On the arithmetic normality of the extended Grassmann variety. Bull. Kyoto Gakugei Univ. Ser. B **13** (1958), 4–6. (Japanese. English summary)

Let L^r and L^s be two linear varieties in a projective space L^n such that $L^r \subset L^s$. Let x, y be, respectively, the Chow-points of L^r, L^s, and V be the set of points (x, y) in a double projective space for a fixed pair (r, s) of integers. Then V is a non-singular variety. The author seems to claim that there is a fixed positive divisor X on V such that for any positive V-divisor Y, there is a positive integer m with $Y \sim mX$. A simple counter example seems to exist for this statement (take, for instance, $n=2$, $r=0$, $s=1$). *T. Matsusaka* (Evanston, Ill.)

2656:

Barsotti, Iacopo. Repartitions on abelian varieties. Illinois J. Math. **2** (1958), 43–70.

Let V be a locally normal irreducible variety defined over a field k. Denote by $k(V)$ its function field, and by $\mathfrak{o}(X, V)$ the local ring of the subvariety X of V. A class of repartitions on V is an element of the direct sum $\sum_X k(V)/\mathfrak{o}(X, V)$, where X ranges over the set of all irreducible subvarieties of codimension 1. The poles of a repartition $\mathfrak{a}=\sum_X \mathfrak{a}_X$ are the subvarieties X for which $\mathfrak{a}_X \neq 0$. The image of $k(V)$ under the natural mapping $\mathrm{cl}_V : k(V) \to \sum_X k(V)/\mathfrak{o}(X, V)$ is denoted by $\mathfrak{B}_s(V)$; its elements are called the exact classes; two classes of repartitions are said to be linearly equivalent if their difference is exact. A class of repartitions $\mathfrak{a}$ is said to be closed if, for every point P of V, there exists an exact class $\mathfrak{b}_P$ such that the poles of $\mathfrak{a}-\mathfrak{b}_P$ do not contain P; the closed classes form a $k(V)$-vector space $\mathfrak{B}(V)$ with a pth power operation (p: characteristic exponent of k). Denoting by $\mathfrak{O}$ the sheaf of local rings on V, and by $\mathfrak{K}$ the constant sheaf with $k(V)$ as stalk, $\mathfrak{B}(V)$ is naturally isomorphic to $H^0(V, \mathfrak{K}/\mathfrak{O})$, and $\mathfrak{B}(V)/\mathfrak{B}_s(V)$ to $H^1(V, \mathfrak{O})$.

Let U be another locally normal variety, and $U \to V$ a regular mapping such that $[k(U) : k(V)]$ is finite; definition and study of the cotrace of a class of repartitions on V, and of the trace of a class of repartitions on U. Behaviour of repartition classes under products and ground field extension.

Let A be an abelian variety. Then $[\mathfrak{B}(A)/\mathfrak{B}_s(A) : k] = \dim(A)$. A class of repartitions is closed if and only if it is semi-invariant (i.e., linearly equivalent to all its translates). A differential class on A is a linear mapping of the k-vector space $\mathfrak{D}_0(A)$ of invariant derivations on A into $\mathfrak{B}(A)$; to every differential ω on A we associate the differential class $\mathrm{cl}(\omega)$ defined by $(\mathrm{cl}(\omega))D = \mathrm{cl}_A(\omega D)$ for every $D \in \mathfrak{D}_0(A)$. Let $X = \sum_i n_i X_i$ (X_i irreducible, $n_i \neq 0$) be a divisor on A; take $x \in k(A)$ such that $v_i(x) = n_i$ for all i (v_i: normed valuation of $k(A)$ centered at X_i); for $D \in \mathfrak{D}_0(A)$, consider the (closed) class of repartitions $\mathfrak{a}(D)$ whose components are the class of $x^{-1}Dx$ in $k(V)/\mathfrak{o}(X_i, V)$ (all i) and 0 elsewhere; the differential class dX such that $(dX)D = \mathfrak{a}(D)$ for every $D \in \mathfrak{D}_0(A)$ is independent of the choice of x; computation rules for the classes dX. If X is algebraically (or numerically; this is the same thing on an abelian variety) equivalent to zero, then there exists a closed differential ω such that $dX = \mathrm{cl}(\omega)$; the converse is true in characteristic 0. Study of the mapping dX in case A is a Jacobian variety and X its theta-divisor. Various applications to the theory of curves, and to the theory of abelian varieties in characteristic $p \neq 0$, are given. *P. Samuel* (Urbana, Ill.)

LINEAR ALGEBRA

See also 2728, 2923, 3040, 3091, 3221.

2657:

★**Neiss, Fritz. Determinanten und Matrizen. 5te Aufl.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vii + 111 pp. DM 6.60.

The third edition, of 1948, was reviewed in MR **11**, 153. The present edition is not essentially different.

2658:

Siddiqi, Omar Ali; and Kulshrestha, P. K. On existence and representation of the Hadamard matric function in the n-dimensional unit space. Ganita **9** (1958), 9–14.

2659:

Gołąb, S. Sur l'équation $f(X)\cdot f(Y)=f(X\cdot Y)$. Ann. Polon. Math. **6** (1959/60), 1–13.

Let E be the set of 2×2 matrices of complex numbers, and suppose f is a mapping from E into C, the complex numbers (f is a functional), such that $f(AB)=f(A)f(B)$ for every pair A, B in E. Then either $f \equiv 0$, or $f(M) \equiv \Phi(\det M)$, where Φ is a complex-valued function such that $\Phi(xy)=\Phi(x)\Phi(y)$. This paper was written in 1956. {The conclusion remains true if the hypotheses are considerably lightened and generalized. Indeed f need not be defined for singular matrices (its definition can then be extended to singular matrices in only one way) and the dimension can be arbitrary.} *J. L. Brenner* (Palo Alto, Calif.)

2660:

Pearl, Martin. On a theorem of M. Riesz. J. Res. Nat. Bur. Standards **62** (1959), 89–94.

The author deals with the problem of determining all 4×4 real skew-symmetric matrices which are orthogonal. Let $\mathfrak{Q}_0 = R+iR+jR+kR$ be the real quaternions, and let $a\to R(a)$, $a\to S(a)$ be the right and left regular representations of the quaternions with respect to the base $(1, i, j, k)$. It is noticed that $R(a)$, $S(a)$ are 4×4 real skew-symmetric matrices and every skew-symmetric matrix Q can be expressed uniquely as a sum $R(a)+S(b)$ for two quaternions a, b. It is then shown that Q is orthogonal if and only if $Q=R(a)$ or $Q=S(b)$; from which follows a theorem due to M. Riesz that if P, Q are two 4×4 skew-symmetric orthogonal matrices then $PQ-QP$ is a multiple of a matrix of the same type. Another result obtained is that every matrix Q of the above type is a unique linear combination of two skew-symmetric, orthogonal matrices. If one replaces the Euclidean metric of the 4-dimensional real space by the Lorentzian metric, then the author shows that there are no 4×4 real skew-symmetric matrices which are orthogonal in this metric.

S. A. Amitsur (New Haven, Conn.)

2661:

Bottema, O. Zur Stabilitätsfrage bei Eigenwertproblemen. Nederl. Akad. Wetensch. Proc. Ser. A. **61** = Indag. Math. **20** (1958), 501–504.

The author deals with the method of H. R. Schwarz [Z. Angew. Math. Phys. **7** (1956), 473–500; MR **18**, 676] for solving the stability problem. H. R. Schwarz first shows how to transform the given matrix A (which may have complex elements) into

$$B=\begin{bmatrix} ib_n' & b_n & & & & \\ -1 & ib_{n-1}' & b_{n-1} & & \mathbf{0} & \\ & -1 & \cdot & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & -1 & ib_2' & b_2 \\ & \mathbf{0} & & & -1 & ib_1'+b_1 \end{bmatrix}$$

where all b's are real, then he applies stability criteria which he has derived from theorems of S. Wall and E. Frank. The author gives a direct and elementary proof of these stability criteria; in addition he states some theorems for the case that some of the superdiagonal b_k in the matrix B vanish. *H. Rutishauser* (Zürich)

2662:

Mirsky, L. Remarks on an existence theorem in matrix theory due to A. Horn. Monatsh. Math. **63** (1959), 241–243.

Two theorems of A. Horn [Proc. Amer. Math. Soc. **5** (1954), 4–7; MR **15**, 847] give solutions to the following two problems. (A) To determine necessary and sufficient conditions for the existence of a matrix with prescribed eigenvalues and singular values. (B) To determine necessary and sufficient conditions for the existence of a positive definite Hermitian matrix with prescribed eigenvalues and prescribed values of the successive upper left principal minors. The present paper offers another proof of these two theorems. In Horn's paper, the theorem concerning (A) was first proved, and then the theorem concerning (B) was derived from that on (A). This logical order is reversed in the present treatment.

Ky Fan (Notre Dame, Ind.)

2663:

Osborn, Howard A. A class of bilinear forms. Trans. Amer. Math. Soc. **90** (1959), 485–498.

Let E be a vector space of dimension n over a field k of characteristic $\neq 2$. For a given linear transformation A of E, let $\mathscr{C}(A)$ denote the set of all bilinear forms f on $E\times E$ such that A is self adjoint with respect to f. That is: $f(Au, v)=f(u, Av)$. In this paper the structure of $\mathscr{C}(A)$ is completely determined and it is shown that $\mathscr{C}(A)$ is isomorphic with $\mathscr{Z}(A)$, where $\mathscr{Z}(A)$ is the centralizer of A in the ring of all linear transformations of E. Furthermore, this isomorphism is a $\mathscr{Z}(A)$-module isomorphism with $\mathscr{Z}(A)$ considered as a right $\mathscr{Z}(A)$-module (using right multiplication) and $\mathscr{C}(A)$ as a right $\mathscr{Z}(A)$-module by setting $(fB)(u, v)=f(u, Bv)$ for $B\in\mathscr{Z}(A)$. Let $\mathscr{S}(A)$ [$\mathscr{A}(A)$] be the space of all bilinear forms f such that both f and fA are symmetric [alternative]; then $\mathscr{C}(A)=\mathscr{S}(A)\oplus_k\mathscr{A}(A)$. If $\mathscr{S}^z(A)$, $\mathscr{A}^z(A)$ are the isomorphic images of $\mathscr{S}(A)$, $\mathscr{A}(A)$ in $\mathscr{Z}(A)$, then with the aid of the ring operation in $\mathscr{Z}(A)$ they become respectively a Jordan and Lie $\mathfrak{A}$-algebra, where $\mathfrak{A}$ is the ring of all linear transformations generated by A. Let $E=E_1\oplus\cdots\oplus E_t$ be the decomposition of E into a direct sum of cyclic A-modules E_i such that the order ideals $\mathfrak{d}(E_i)$ satisfy: $\mathfrak{d}(E_i)\supseteq\mathfrak{d}(E_j)$ if $i\leq j$. The author gives a constructive method to obtain the forms of $\mathscr{S}(A)$ and $\mathscr{A}(A)$, by which he shows that: (1) $\dim_k\mathscr{S}(A)=\sum_{j=1}^{t}(t-j+1)\dim_k E_j$; (2) $\dim_k\mathscr{A}(A)=\sum_{j=1}^{t}(t-j)\dim_k E_j$; (3)

$$\dim_k\mathscr{C}(A)=\sum_{j=1}^{t}(2t-2j+1)\dim_k E_j.$$

This yields the known results on the dimension of the centralizer $\mathscr{Z}(A)$ of A. The author gives also an interesting matrix representation of $\mathscr{C}(A)$ which is too long to be described here. *S. A. Amitsur* (New Haven, Conn.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 2663, 2693, 2696, 2982.

2664:

Barsotti, Iacopo. Recenti sviluppi dell'algebra. I. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 49–55.

This is the first in a series of expository papers based on lectures given by the author at various Italian universities in 1958. This one is devoted to the theory of commutative rings. It includes an account, without proofs, of the ideal theory of Noetherian and Dedekind rings. Some of the applications to algebraic geometry are indicated, in particular the notions of normal varieties and of linear equivalence. *W. E. Jenner* (Lewisburg, Pa.)

2665:

Barsotti, Iacopo. Recenti sviluppi dell'algebra. II. Rend. Sem. Mat. Fis. Milano **28** (1959), 138–145. (English summary)

This is the second of two expository articles on the theory of commutative rings [cf. the above review]. This one is concerned with valuation theory. There are many well chosen examples and some of the applications to

algebraic geometry are indicated. The advantages of valuation theory over purely ideal-theoretic methods in some situations are commendably set forth.

W. E. Jenner (Lewisburg, Pa.)

2666:

★**Lesieur, L.; et Croisot, R. Théorie Noethérienne des anneaux, des demi-groupes et des modules dans le cas non commutatif. I.** Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 79–121. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

The results of this paper have been announced previously by the authors. [See C. R. Acad. Sci. Paris **243** (1956), 1988–1991; MR **18**, 637.] The proofs, published here for the first time, are lattice-theoretic in nature and apply not only to ideals in rings but also to ideals in a demigroup and to submodules of a module over a non-commutative ring. *D. C. Murdoch* (Vancouver, B.C.)

2667:

Lesieur, L.; et Croisot, R. Théorie Noethérienne des anneaux, des demi-groupes et des modules dans le cas non commutatif. II. Math. Ann. **134** (1958), 458–476.

The results obtained in this paper are essentially the same as those in the first paper of the series [see above review]. The methods used are quite different, the main tool being the concept of the tertiary radical of an ideal. Let U be any ring, or demigroup with 0 element, in which either the ascending chain condition (ACC) or descending chain condition (DCC) for left ideals is satisfied. Let a and b be elements of U and denote by aU^*b the union of the sets aUb and $\{ab\}$. If A is a left ideal of U, the two-sided ideal $R(A)$ consists of all elements a such that for every $b \notin A$, there exists x in the left ideal generated by b such that $x \notin A$ and $aU^*x \subseteq A$. This ideal $R(A)$ is called the tertiary radical of A. A tertiary left ideal can then be defined as a left ideal T such that $aU^*b \subseteq T$ and $b \notin T$ together imply $a \in R(T)$. This definition is equivalent to that given in the first paper of the series. It then follows that the tertiary radical of a tertiary left ideal is a two-sided prime ideal, and every intersection-irreducible left ideal is tertiary. This leads to the usual existence and uniqueness theorems for the representation of any left ideal as the intersection of tertiary left ideals. A similar theory is then constructed for two-sided ideals in U and then for the submodules of a module over a non-commutative ring. If the DCC holds for left ideals, the tertiary radical of a two-sided ideal coincides with its tertiary radical when it is considered as a left ideal only. Other consequences of the DCC announced previously [C. R. Acad. Sci. Paris, loc. cit., above review] are also obtained. Several illustrative examples are discussed.

D. C. Murdoch (Vancouver, B.C.)

2668:

Lesieur, L.; et Croisot, R. Théorie Noethérienne des anneaux, des demi-groupes et des modules dans le cas non commutatif. III: Sur la notion de radical. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 75–93.

In this third paper of the series the authors return to the lattice-theoretic setting of the first paper. Let $(\mathscr{L})$ be a lattice, whose elements will be denoted by script letters, in which a multiplication is defined and whose elements act as left operators on a second lattice (L) whose elements are denoted by Latin capital letters. For every element X in (L) there is a right residual by each element of $(\mathscr{L})$, which is an element of (L), and a left residual by each element of (L), which is an element of $(\mathscr{L})$. The set of left residuals and the set of right residuals of any element X of (L) are assumed to satisfy the ACC. The application to rings is obtained by taking $(\mathscr{L})$ to be the lattice of two-sided ideals and (L) to be that of the left ideals.

For any element X of (L) five different radicals are defined. The primary radical $\mathscr{R}_1(X)$ is the classical radical. The secondary radical $\mathscr{R}_2(X)$ is related to the secondary elements of (L) as $\mathscr{R}_1(X)$ is to the primary elements. The tertiary radical $\mathscr{R}_3(X)$, which has been defined in the previous review, is here given a lattice-theoretic definition. The "radical unirésidué", denoted by $\mathscr{R}'(X)$, is the intersection of the prime proper left residuals of X. Finally the primal radical $\mathscr{R}_4(X)$ is the intersection of the maximal proper left residuals of X and coincides in the case of a two-sided ideal with the adjoint of X in the sense of C. W. Curtis [Amer. J. Math. **74** (1952), 687–700; MR **14**, 127]. The five radicals are ordered thus:

$$\mathscr{R}_1(X) \subseteq \mathscr{R}_2(X) \subseteq \mathscr{R}'(X) \subseteq \mathscr{R}_3(X) \subseteq \mathscr{R}_4(X).$$

In the commutative case $\mathscr{R}_1(X) = \mathscr{R}_2(X) = \mathscr{R}'(X)$ and each of these is the classical radical.

D. C. Murdoch (Vancouver, B.C.)

2669:

Auslander, Maurice; and Buchsbaum, D. A. Unique factorization in regular local rings. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 733–734.

The unique factorization in regular local rings was an open question: It was known that if every regular local ring of rank (=Krull dimension) 3 is a unique factorization ring then so is every regular local ring [see the reviewer's paper, Amer. J. Math. **80** (1958), 382–420; MR **20** #862]. The authors prove in this paper that every regular local ring of rank 3 is a unique factorization ring, hence they give the final answer to the problem.

The proof is strikingly simple and depends on the following simple lemma: Let R be a local ring with maximal ideal $\mathfrak{M}$ and let $x \in \mathfrak{M}$ and $y \in R$ be such that, for $\mathfrak{a} = (x):y$, (a) homological dimension of $\mathfrak{a}$ is at most 1 and (b) x is not in $\mathfrak{M}\mathfrak{a}$. Then $\mathfrak{a} = (x)$ (and x is not a zero divisor).

The authors prove the unique factorization in the following form: If a local domain R is such that rank $R \leq 3$, hd $R/\mathfrak{p} < \infty$ for all minimal prime ideals $\mathfrak{p}$, then R is a unique factorization ring. *M. Nagata* (Kyoto)

2670:

Assmus, E. F., Jr. On the homology of local rings. Illinois J. Math. **3** (1959), 187–199.

Let R be a local, noetherian, commutative ring with unit and residue class field K. It is well known that $T = \mathrm{Tor}^R(K, K)$ has a natural multiplication which makes it a graded K-algebra. $T \otimes_K T$ is considered a K-algebra in the usual way the tensor product of two graded algebras is considered an algebra. The author introduces a graded map $M: T \to T \otimes_K T$ of degree 0 which has the following properties: (a) M is a K-algebra homomorphism; (b) if x

is in $\mathrm{Tor}_n^R(K,K)$, then $M(x)=x\otimes 1+y+1\otimes x$ for some y in

$$\sum_{i+j=n}\mathrm{Tor}_i^R(K,K)\otimes_K\mathrm{Tor}_j^R(K,K)$$

where $i\neq n$ and $j\neq n$. That is, it is shown that $\mathrm{Tor}^R(K,K)$ is a Hopf algebra. This fact is then used to give another proof of Serre's result that $\dim_K \mathrm{Tor}_p^R(K,K)\geq\binom{n}{p}$ where n is the minimum number of elements required to generate the maximal ideal of R [see J. P. Serre, Proc. Internat. Sympos. Algebraic Number Theory, Tokyo and Nikko, 1955, pp. 175–189, Science Council of Japan, Tokyo, 1956; MR **19**, 119]. The paper concludes with a homological characterization of local rings which are complete intersections in terms of the multiplication which can be introduced in the homology of the Koszul complex generated by a minimal generating set for the maximal ideal of the local ring. *M. Auslander* (Waltham, Mass.)

2671:

Northcott, D. G. An algebraic relation connected with the theory of curves on non-singular surfaces. J. London Math. Soc. **34** (1959), 195–204.

Let Q be a one-dimensional local ring which is the factor ring of a two-dimensional regular local ring by an unmixed (i.e., principal) ideal, e.g., the local ring of a point P on a curve C that lies on a surface S on which P is simple. The conductor from Q to its first neighbourhood ring R [Northcott, Proc. London Math. Soc. (3) **8** (1958), 388–415; MR **20** #3133] is $\mathfrak{m}^{e-1}$, where $\mathfrak{m}$ is the maximal ideal of Q and e its multiplicity; the length $l_Q(R/Q)$ is $\frac{1}{2}e(e-1)$. From this one deduces that, if Q' is the integral closure of Q in its total quotient ring, then

$$2l_Q(Q'/Q)=e(e-1)+\sum{}^{(1)}e_1(e_1-1)[K_1:K]+\cdots$$
$$+\sum{}^{(n)}e_n(e_n-1)[K_n:K]+\cdots,$$

where the sum $\sum^{(n)}$ is extended to all the local rings Q_n in the nth neighbourhood of Q [loc. cit.], and where e_n denotes the multiplicity of Q_n, K_n its residue field, and K the residue field of Q. In the geometric case the above formula gives (twice) the reducing effect of a singularity on the genus of a curve that lies on a non-singular surface. *P. Samuel* (Urbana, Ill.)

2672:

Williamson, J. H. On the functional representation of certain algebraic systems. Pacific J. Math. **7** (1957), 1251–1277.

Let A be a commutative algebra over a field K, and let B be a subalgebra with unit. Let J be a B-submodule of A. For $a\in A$, define $\hat{a}(J)$ to equal λ if some λ in K exists such that $a\in\lambda+J$. Otherwise, set $\hat{a}(J)=\infty$, where ∞ is a symbol, which, together with K gives an algebraic system K', with $\lambda\cdot\infty$ defined as ∞ for $\lambda\neq 0$, $\lambda+\infty$ defined as ∞ for $\lambda\neq\infty$. Let $\mathfrak{J}$ [$\mathfrak{J}'$] be the set of B-modules [of B-modules that are algebras], and let $\mathfrak{J}_0$ [$\mathfrak{J}_0'$] be the set of maximal elements in $\mathfrak{J}$ [$\mathfrak{J}'$] which are subalgebras. The representation $a\to\hat{a}$, of A by K'-valued functions on $\mathfrak{J}_0$ [$\mathfrak{J}_0'$], is studied. The paper is devoted almost wholly to algebraic matters; e.g., what is the effect of algebraic closedness in K, and what are the kernels of the representations? Topological aspects of the theory are to be treated elsewhere. *R. Arens* (Los Angeles, Calif.)

2673:

Curtis, Charles W. Modules whose annihilators are direct summands. Pacific J. Math. **8** (1958), 685–691.

Let $B=\sum_{s\in G}\Delta b_s$ be a crossed product of a finite group G and a division ring Δ (with respect to a certain homomorphism of G onto a group of automorphisms of Δ, and having a certain factor set). With a right B-module M the equivalence of the following three conditions is proved: (a) the annihilator ideal of M in B is a direct summand of B; (b) for every indecomposable central idempotent ε of B, the submodule $M\varepsilon$ of M is either 0 or contains a submodule isomorphic to the direct sum of all non-isomorphic indecomposable right-ideal direct summands of $B\varepsilon$; (c) (M', M, τ_f) is a regular pairing in the sense of the author [Canad. J. Math. **8** (1956), 271–292; MR **18**, 187], where M' is a left B-module paired with M to B by a non-degenerate bilinear form f, and τ_f is defined by $\tau_f(\psi,x)=\sum_{s\in G}f(\psi,xb_s)b_s^{-1}$. *T. Nakayama* (Nagoya)

2674:

Nakano, Takeo. A nearly semi-simple ring. Comment. Math. Univ. St. Paul. **7** (1959), 27–33.

Let I be a left ideal in an arbitrary ring R with $I^2\neq 0$. Then $\mathrm{Hom}_R(I,I)$ is a division ring if and only if for each a in I, $Ia=I$ or 0. In this case $I=Re$ with e an indecomposable idempotent of R. Left ideals with these properties are called quasi-minimal. If (*) every principal left ideal in R is projective, the quasi-minimal ideals are all those left ideals whose proper subleft ideals are nil. Under hypothesis (*) on R two quasi-minimal left ideals I, I' are isomorphic R-modules if and only if both $\mathrm{Hom}_R(I,I')$ and $\mathrm{Hom}_R(I',I)$ are not zero. It is then possible to introduce a partial ordering into the set of quasi-minimal left-ideals in a ring satisfying (*). Finally, the author calls a ring a P-ring if R satisfies (*) and $R=I_1\oplus\cdots\oplus I_n$ with I_i quasi-minimal. He shows that R is isomorphic to the ring $\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}$ where A is a P-ring with a smaller n, $C\neq 0$ is a simple ring with minimum condition, and B is a left A-, right C-module such that every cyclic A-submodule is A-projective. Conversely, any such ring of triangular matrices is a P-ring. These results are related to those obtained for semi-primary rings of global dimension one by Nakayama and Jans [Nagoya Math. J. **11** (1957), 67–76; MR **19**, 250]. *A. Rosenberg* (Evanston, Ill.)

2675:

Amitsur, S. A. The radical of field extensions. Bull. Res. Council Israel. Sect. F **7F** (1957/58), 1–10.

Let A be an algebra over a field C with Jacobson radical N. The author first proves that if F is a (finite or infinite) separable algebraic extension field of C, then the radical of $A\otimes_C F$ is $N\otimes_C F$. The proof is quite long and is carried out by reduction to the case of $[F:C]<\infty$. In this case, however, the theorem is an immediate consequence of theorems 5.8.1 and 5.14.1 in N. Jacobson, *Structure of rings* [Amer. Math. Soc. Colloq. Publ., Providence, R.I., 1956; MR **18**, 373]. Also, Villamayor [Pacific. J. Math. **9** (1959), 941–951] proves a generalization of this theorem with homological methods which yield a briefer proof. Next, the case where F is a purely transcendental extension field of C is studied. In that case the radical of $A\otimes_C F$ is $J\otimes_C F$ where J is a nil-ideal of A.

There are also some applications of this result to the LBI radicals of A and $A \otimes_C F$ introduced by the author in Proc. Amer. Math. Soc. **7** (1956), 35–48 [MR **17**, 822]. Finally, these results are applied to the problem of the semisimplicity of the group algebra of an infinite group over any field of characteristic zero. It is shown that in order to solve this problem it is sufficient to prove that group algebras over the rationals are semisimple. Furthermore, the author shows that group algebras over fields of infinite transcendence degree over the rationals are semisimple. Thus, in particular, the problem is solved for any non-denumerable field of characteristic zero. The author points out that this last was also done by I. N. Herstein (unpublished). *A. Rosenberg* (Evanston, Ill.)

2676:

Farahat, H. K.; and Mirsky, L. Group membership in rings of various types. Math. Z. **70** (1958), 231–244.

The authors consider conditions under which an element of a ring is an element of some subgroup of the multiplicative semigroup of the ring, continuing the work of A. Ramun [Amer. J. Math. **31** (1909), 18–41; **49** (1927), 285–308]. The authors do not mention a paper of their own [Amer. Math. Monthly **63** (1956), 410–412; MR **18**, 4] dealing with group-membership in rings of matrices. As shown by Kimura [Kōdai Math. Sem. Rep. **1954**, 85–88; MR **16**, 443], each idempotent element e of a semigroup S is contained in a maximal subgroup $\mathfrak{M}(e)$ of S, and if $e \neq f = f^2$ then $\mathfrak{M}(e)$ and $\mathfrak{M}(f)$ are disjoint. [See also Š. Schwarz, Sb. Prác. Prirod. Fak. Slovensk. Univ. Bratislave no. 6 (1943); MR **10**, 12.] If R is a ring with identity element, and $\mathfrak{C}(e)$ is the set of all invertible elements of R commuting with e, then $\mathfrak{M}(e) = e\mathfrak{C}(e)$; this theorem does not hold in general for semigroups.

Certain fairly elementary results are obtained concerning maximal subgroups of the ring of integers mod n (n a positive integer); of a principal ideal ring modulo a non-zero ideal; and of a direct sum of rings R_1 (called divided rings) having identity elements and such that each element of R_1 is either invertible or nilpotent. Results, similar to those obtained independently by W. E. Barnes and H. Schneider [Arch. Math. **8** (1957), 166–168; MR **19**, 1155], are obtained for linear associative algebras over a field. *D. W. Wall* (Ann Arbor, Mich.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

2677:

Schafer, R. D. On non-commutative Jordan algebras. Proc. Amer. Math. Soc. **9** (1958), 110–117.

In case the algebras designated by the title are of characteristic 0, the author showed [Proc. Amer. Math. Soc. **6** (1955), 472–475; MR **17**, 10] that they are trace-admissible and thereby possess a bilinear form capable of distinguishing the radical and reducing the classification of semi-simple algebras to the simple case. A division of the simple algebras into several classes is also obtained from earlier results on trace-admissible algebras. In this paper he extends this type of analysis and its results to algebras of characteristic $\neq 2$, provided that certain irregular algebras of degree 1, which cannot be commutative, do not intrude. Examples of these "nodal" algebras show that this restriction is real. Simple nodal algebras have dimensions divisible by the characteristic. With these exceptions, however, the non-commutative Jordan algebras inherit an admissible trace function from their associated symmetrized commutative algebras and may therefore be attacked by the method used in the characteristic 0 case. *W. G. Lister* (Oyster Bay, N.Y.)

2678:

Schafer, R. D. Restricted noncommutative Jordan algebras of characteristic p. Proc. Amer. Math. Soc. **9** (1958), 141–144.

In the paper reviewed above the author established structure theorems by proving trace-admissibility, a process which cannot be applied to the nodal algebras. He now devises an identity which implies trace-admissibility for noncommutative Jordan algebras of characteristic $p > 2$ and which defines the class of algebras denoted by the title. If R_x is right multiplication by x, the identity is $(R_x)^p = R_{x^p}$. These "restricted" algebras cannot be nodal and the trace method works. Since not all of the simple, trace-admissible algebras are restricted, a suitable modification of the classification of the simple algebras is made. *W. G. Lister* (Oyster Bay, N.Y.)

2679:

Leger, G. F. A note on derivations of Lie algebras. II. Proc. Amer. Math. Soc. **10** (1959), 10–11.

Let L be a Lie algebra of characteristic 0 with radical R, Levi decomposition $L = S + R$, and derivation algebra $D(L)$. When R is regarded as an S-module, denote the trivial submodule of R by R^S. Theorem: If R^S is contained in the center of R, then $D(L)$ splits over its ideal $I(L)$ of all inner derivations of L. This implies that the example given in the author's earlier paper [same Proc. **4** (1953), 511–514; MR **15**, 6] is an algebra L such that $D(L)$ splits over $I(L)$ but $D(R)$ does not split over $I(R)$.

R. D. Schafer (Cambridge, Mass.)

HOMOLOGICAL ALGEBRA

See also 2670, 2674, 2675.

2680a:

Bockstein, Meyer. Sur le spectre d'homologie d'un complexe. C. R. Acad. Sci. Paris **247** (1958), 259–261.

2680b:

Bockstein, Meyer. Sur la formule des coefficients universels pour les groupes d'homologie. C. R. Acad. Sci. Paris **247** (1958), 396–398.

Let L be a cochain complex of Abelian groups, G any Abelian group and $H^q(L, G)$ the qth homology group of the complex $L \otimes G$. Let $\{H^q(L, Z_m); \pi, \omega\}$, $m = 0, 1, 2, \cdots$, be the homology spectrum of L [see M. Bokšteĭn, C. R. (Doklady) Acad. Sci. URSS (N.S.) **37** (1942), 243–245; MR **5**, 48], let ${}_mG = \{g \mid g \in G,\ mg = 0\}$, and let $i = i_{m'}^{m}$: ${}_mG \to {}_{m'}G$ and $j = j_m^{m'}$: ${}_{m'}G \to {}_mG$ be the obvious homomorphisms of injection and multiplication by m'/m,

defined whenever m divides m'. The homology spectrum and $\{{}_mG; i, j\}$ form a conjugate pair of systems of groups in the sense of the following definition. $\{G_\alpha; \pi\}$ and $\{H_\alpha; \omega\}$ are two conjugate systems if G_α and H_α are Abelian groups defined for the same set of indices α, and $\pi = \pi_\beta^\alpha$: $G_\alpha \to G_\beta$, $\omega = \omega_\alpha^\beta$: $H_\beta \to H_\alpha$ are homomorphisms defined for the same set of pairs (α, β). A tensor product $\{G_\alpha; \pi\} \otimes \{H_\alpha; \omega\}$ of two conjugate systems is defined as the quotient group of the direct sum $\sum_\alpha G_\alpha \otimes H_\alpha$ by the subgroup generated by elements of the form $g_\alpha \otimes \omega_\alpha^\beta h_\beta - \pi_\beta^\alpha g_\alpha \otimes h_\beta$, $g_\alpha \in G_\alpha$, $h_\beta \in H_\beta$.

The main result of the first paper asserts that

$$H^q(L, G) \approx \{H^q(L, Z_m); \pi, \omega\} \otimes \{{}_mG; i, j\}$$

whenever L consists of torsion-free groups L^q. Under the same assumptions for L, the author proves in the second paper that the well-known exact sequence

$$0 \to H^q(L, Z) \otimes G \to H^q(L, G) \to \operatorname{Tor}(H^{q+1}(L, Z), G) \to 0$$

splits. The classical case, previously known, is the case of a free complex L [see p. 161 of S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, 1952; MR **14**, 398].

Both papers are algebraic versions of the first and the second half, respectively, of M. Bokšteĭn, Dokl. Akad. Nauk SSSR **119** (1958), 1066–1069 [MR **20** #4263]. Analogous results were obtained there for compact Hausdorff spaces and Čech cohomology, which is in fact the homology of a torsion-free cochain complex. Methods of proof coincide. *S. Mardešić* (Zagreb)

2681:

Bokšteĭn, M. On the Künneth formula in homological algebra. Dokl. Akad. Nauk SSSR **124** (1959), 1187–1190. (Russian)

The author proves the Künneth formula for differential involutive groups without elements of finite order (a differential group is said to be involutive if an automorphism ω is given such that $\omega\omega = 1$ and $d\omega + \omega d = 0$).

Following his previous papers [reviewed above] the author deals with the spectrum $\{H(A), H_m(A); \pi_m{}^0, \pi_m{}^{m'}, \omega_{m'}{}^m, \delta_0{}^m\}$, now called complete because the Bockstein connecting homomorphism $\delta_0{}^m$ is added. Starting from the formula

$$H(K \otimes L) = \{H(K), H_m(K); \pi_m{}^0\omega, \pi_m{}^{m'}, \omega_{m'}{}^m, \delta_0{}^m\} \otimes \{H(L), H_m(L); \delta_0{}^m, \omega_{m'}{}^m, \pi_m{}^{m'}, \pi_m{}^0\}$$

taken from the papers above, one obtains $H(K \otimes L) \cong H(K) \otimes H(L) + \sum H_m(K) \otimes H_m(L)$. Using the relation $H_m(A) \cong [H(A)]_m + {}_m[H(A)]$, it is possible to cancel out some terms, obtaining

$$H(K \otimes L) \cong H(K) \otimes H(L) + \{{}_m[H(K)]; i, j\} \otimes \{{}_m[H(L)]; j, i\}$$

and finally the Künneth formula

$$H(K \otimes L) \cong H(K) \otimes H(L) + H(K) * H(L).$$

In particular, if K and L are complexes without elements of finite order, one may write $H^q(K \otimes L) \cong \sum_{r+r'=q} H^r(K) \otimes H^{r'}(L) + \sum_{r+r'=q-1} H^r(K) * H^{r'}(L)$.

E. Lluis (Mexico, D. F.)

GROUPS AND GENERALIZATIONS

See also 2666, 2667, 2668, 2675, 2955.

2682:

Petresco, Julian. Sur le théorème de Kuroš dans les produits libres. Ann. Sci. École Norm. Sup. (3) **75** (1958), 107–123.

The purpose of this paper is to carry the procedures of the author in an earlier paper on free groups [Bull. Sci. Math. (2) **80** (1956), 6–32; MR **18**, 716] over to free products. This results in the longest proof of the theorem of Kurosch published so far. Unfortunately it is difficult for the reader to ascertain whether the subgroup H of the free product is exhibited as the free product of the groups B_β which he exhibits or is merely their union.

Marshall Hall, Jr. (Pasadena, Calif.)

2683:

Yacoub, K. R. On the general products of two finite cyclic groups one of which being order p^2. Publ. Math. Debrecen **6** (1959), 26–39.

The group G is the general product or Zappa-Szép-Rédei product of its subgroups A and B if $AB = G$ and $A \cap B = \{1\}$. The author continues his study of the case that A and B are finite cyclic groups, begun in his London Ph.D. thesis (1953), and the subject of his papers in Proc. Glasgow Math. Assoc. **2** (1955), 116–123; **3** (1956), 18–35; **3** (1958), 164–169; **4** (1959), 3–6; and Duke Math. J. **24** (1957), 455–465 [MR **17**, 11; **19**, 5; **20** #5235c; **19**, 1037]. The present instalment attacks the case that the order of B is the square of an odd prime. All groups of this kind can be specified by a small number of arithmetical invariants; however, the question when groups with different representations as general products, and thus with possibly different invariants, are isomorphic is not completely answered. The method, though elementary, is highly involved. {By more powerful and refined methods, B. Huppert [Math. Z. **58** (1953), 243–264; MR **14**, 1059] has obtained deeper results about the structure of the product of two finite cyclic groups (not necessarily intersecting in the trivial group); there is no overlap between Huppert's work and that of the author.}

B. H. Neumann (Bombay)

2684:

Łoś, J. Linear equations and pure subgroups. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 13–18.

For sequences of integers of type ω_λ, the first ordinal of the power of the cardinal $\aleph_\lambda$, construct the group $S_\lambda{}^*$ of all such sequences. The author shows that if $\aleph_\lambda$ is of measure 0, then no direct summand of $S_\lambda{}^*$ can include S_λ, the free abelian group with $\aleph_\lambda$ generators. For an ordinal κ, let $S_\lambda{}^\kappa$ be the subgroup of those sequences, the number of non-zero entries of each of which is below $\aleph_\kappa$. Then, if $\aleph_\kappa$ is regular, $S_\lambda{}^\kappa$ is κ-pure in $S_\lambda{}^*$ in the sense that every system of m linear equations in m unknowns, $m < \aleph_\kappa$, each with integral coefficients almost all zero, with right members in $S_\lambda{}^\kappa$, which has a set of solutions in $S_\lambda{}^*$, has also at least one set of solutions in $S_\lambda{}^\kappa$. Let Z be the additive group of integers. An abelian group G is called regular if $\bigcap \ker \phi$ is trivial where the intersection is taken over all $\phi \in \operatorname{Hom}(G, Z)$. If $\aleph_\kappa$ is a regular aleph, if $G \subset S_\lambda{}^*/S_\lambda{}^\kappa$, and if $|G| < \aleph_\kappa$, then G is regular. Mapping

all the constant sequences in $S_\lambda{}^*$ into $S_\lambda{}^*/S_\lambda{}^\kappa$, the author finds that the resulting subgroup is κ-pure in this quotient whenever $\aleph_\kappa$ is regular. More generally, every pure cyclic subgroup of this quotient is κ-pure. For $\aleph_\kappa$ also of measure 0, every direct summand of this quotient has order $\geq \aleph_\kappa$. Under these conditions, for a κ-compact space (every family of closed subsets with void intersection contains a subfamily of cardinality $< \aleph_\kappa$ with void intersection) which is the product of $2^{\aleph_\kappa}$ metric spaces, at least one of these factors is compact. *F. Haimo* (St. Louis, Mo.)

2685:

Łoś, J. Generalized limits in algebraically compact groups. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 19–21.

Let G be an abelian group and λ be a cardinal number. By a generalized λ-limit on G, the author means a function which gives to each sequence $\{x_\xi\}_{\xi<\omega_\lambda}$ of type ω_λ (where ω_λ is the first ordinal of power λ) an element $\mathrm{Lim}_{\xi<\omega_\lambda} x_\xi$ in G, such that: (1) $\mathrm{Lim}_{\xi<\omega_\lambda}(x_\xi+y_\xi)=\mathrm{Lim}_{\xi<\omega_\lambda} x_\xi+\mathrm{Lim}_{\xi<\omega_\lambda} y_\xi$; (2) if $x_\xi=g$ for $\xi<\omega_\lambda$, then $\mathrm{Lim}_{\xi<\omega_\lambda} x_\xi=g$; (3) if there exists an ordinal $\xi_0<\omega_\lambda$ such that $x_\xi=y_\xi$ for $\xi\geq\xi_0$, then $\mathrm{Lim}_{\xi<\omega_\lambda} x_\xi=\mathrm{Lim}_{\xi<\omega_\lambda} y_\xi$. He proves that G has a λ-limit for every λ if and only if G is algebraically compact (a direct summand of a direct product of finite cyclic groups and divisible groups). His method is to consider the group of all ω_λ sequences modulo those which are zero after some ξ_0, embed G as a pure subgroup in this, and then prove that G will always be a direct summand here if and only if it is always a direct summand whenever it is a pure subgroup (this is equivalent to being algebraically compact). *D. K. Harrison* (Philadelphia, Pa.)

2686:

Sąsiada, E. Construction of directly indecomposable Abelian groups of power higher than that of the continuum. II. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 23–26.

[For part I, see Bull. Acad. Polon. Sci. Cl. III **5** (1957), 701–703, LXII; MR **19**, 632.] An abelian group is indecomposable if it is not (except trivially) a direct sum of two subgroups. The question of whether there exist indecomposable groups of arbitrarily high cardinality is still open. In this paper the author proves that for $\aleph_1$ distinct cardinals n, there exist indecomposable groups of rank n. The indecomposable groups are actually constructed, but are (it seems, necessarily) somewhat involved. *D. K. Harrison* (Philadelphia, Pa.)

2687:

Baer, Reinhold. Überauflösbare Gruppen. Abh. Math. Sem. Univ. Hamburg **23** (1959), 11–28.

It is proved that the group G is supersoluble, in the sense that every non-trivial homomorphic image has a non-trivial cyclic normal subgroup, and satisfies the maximum condition for subgroups, if and only if (a) G is finitely generated; (b) every maximal subgroup of G has prime index; and (c) every infinite homomorphic image of G has an almost completely reducible finitely generated normal subgroup. (The normal subgroup N of G is almost completely reducible if it is not 1, and the normal subgroups X for which N/X is a finite minimal normal subgroup of G/X have trivial intersection.) An important part is played by the lemma, proved for finite and for abelian G, that an integral representation of G, of finite degree, which is rationally irreducible, and whose reduction modulo p has a component of degree 1 for almost all primes p, is itself of degree 1. *Graham Higman* (Oxford)

2688:

Wielandt, Helmut; und Huppert, Bertram. Normalteiler mehrfach transitiver Permutationsgruppen. Arch. Math. **9** (1958), 18–26.

This paper is devoted to the proof of theorems which are stronger than Jordan's result that, in general, a non-trivial normal subgroup of a k-ply transitive group is $(k-1)$-ply transitive. A permutation group G is called simply primitive if it is primitive in the usual sense, and is called k-ply primitive if it is transitive and a stabiliser G_1 is $(k-1)$-ply primitive. G is called $\frac{1}{2}$-ply transitive if either it is of degree 1, or all its systems of transitivity have the same cardinal greater than 1; and G is called $(k+\frac{1}{2})$-ply transitive if it is transitive and G_1 is $(k-\frac{1}{2})$-ply transitive. Each of k-ple primitivity and $(k+\frac{1}{2})$-ple transitivity has the properties one would expect of a half-way house between k-ple and $(k+1)$-ple transitivity, except that it is an open question whether a group with a $(k+\frac{1}{2})$-ply transitive subgroup is necessarily at least $(k+\frac{1}{2})$-ply transitive.

The main theorem of the paper is that, in general, a non-trivial normal subgroup of a k-ply transitive group is $(k-\frac{1}{2})$-ply transitive, k an integer or half an odd integer. Exceptions occur when G is symmetric, and for certain small values of k. These are all listed explicitly, except that when $k=3/2$, it is only known that G is a Frobenius group. It is also shown that, with certain explicit exceptions, a non-trivial normal subgroup of a k-ply primitive group is k-ply transitive; but it is not known how to make precise the assertion that "in general" a normal subgroup of a k-ply transitive group is $(k-1)$-ply primitive.

As would be expected, proofs are mostly by induction on k. If the normal subgroup N is regular, the question is, how transitive can the automorphism group of a finite group be. If N is not regular, the induction hypothesis applies to the stabiliser N_1 as a normal subgroup of G_1, and the main burden of the proof consists in determining for how many steps of the induction the effect of the exceptions persists. *Graham Higman* (Oxford)

2689:

Itô, Noboru. Normalteiler mehrfach transitiver Permutationsgruppen. Math. Z. **70** (1958), 165–173.

Let G be a k-ply transitive permutation group, but not the symmetric group. Wielandt has called G k-ply primitive if the subgroups of G that leave $k-1$ symbols fixed are primitive on the remaining symbols; and $(k+\frac{1}{2})$-ply transitive if they have domains of transitivity of equal length >1. The classical theorem of Jordan that a proper normal subgroup of G is $(k-1)$-ply transitive (except possibly when $k=3$ and the degree is a power of 2) has recently been refined by Wielandt and Huppert [paper reviewed above] as follows: 1. A non-regular normal subgroup of G is $(k-1)$-ply transitive. 2. If G is $(k+\frac{1}{2})$-ply transitive, then a non-regular normal subgroup is k-ply

transitive (except possibly when $k=1$ or 2). 3. If G is k-ply primitive, then a non-regular normal subgroup is k-ply transitive. The main result of the present paper is: 4. For $k>2$, a non-regular normal subgroup of G is $(k-1)$-ply primitive.—It is quite impossible to give an indication of the intricate way in which the author derives this theorem. Suffice it to say that the proof falls into 13 steps, each closely argued, and fills more than seven pages of solid text. *K. A. Hirsch* (London)

2690:

Wussing, Hans. Über Einbettungen endlicher Gruppen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **103** (1958), no. 3, 38 pp.

It is proved (among other things of a like kind) that, for fixed positive integers m, n, both greater than 1, there are arbitrarily large symmetric groups which can be generated by an element of order m and one of order n, if and only if m, n are neither both odd, nor both 2. It is correctly concluded that if m, n are neither both odd nor both 2, every finite group can be embedded in a finite group with two generators, one of order m and one of order n. The deduction made in the contrary case is, however, false (except when $m=n=2$).

Graham Higman (Oxford)

2691:

Rosati, Luigi Antonio. Sui gruppi finiti a sottogruppi di composizione caratteristici. Rend. Sem. Mat. Univ. Padova **28** (1958), 331–337.

The author considers the following three properties for a group G: (A) any composition subgroup of G is a power of G, (B) any composition subgroup of G is characteristic in G, and (C) if H is any normal subgroup of G, and S any composition subgroup of G, then SH/H is a characteristic subgroup of G/H. The assumption that G is finite and solvable is imposed throughout, and Ph. Hall's characterization of finite solvable groups is used. E. Schenkeman has proved that (A) holds if and only if G is metacyclic and the derived subgroup G' has order prime to its index [Proc. Amer. Math. Soc. **8** (1957), 664–667; MR **19**, 386]. The present author points out that (C) implies (B), and that (B) implies transitivity of normality for subgroups of G (i.e., G is a t-group). He provides an example of a finite solvable group satisfying (B) which is not metacyclic, but shows that (C) and (A) are equivalent.

D. G. Higman (Ann Arbor, Mich.)

2692:

Haber, Seymour; and Rosenfeld, Azriel. Groups as unions of proper subgroups. Amer. Math. Monthly **66** (1959), 491–494.

Let a group G be covered (irredundantly) by n proper subgroups, $G=A_1\cup\cdots\cup A_n$; then in the intersection $A_1\cap\cdots\cap A_n$, every single A_i is redundant. In particular $n>2$. Moreover G admits a representation $A_1\cup A_2\cup A_3$ if and only if the Klein-group of order 4 is homomorphic to G. A representation by n proper subgroups is impossible if every $g\in G$ is divisible by every positive integer $<n$ (equivalently, for finite G, if $(|G|,(n-1)!)=1$). If $n=p+1$, where p is the smallest prime dividing $|G|$, and the union is irredundant, then at least one A_i, say A_1, has the index p; if in particular A_1 is normal, then every A_i has the index p and p^2 divides $|G|$. *F. W. Levi* (Freiburg)

2693:

Szele, Tibor; and Kertész, Andor. On generalized p-groups. Acta Univ. Debrecen. **2** (1955), 131–135. (Hungarian. English summary)

Let Z_p denote the ring of p-adic integers. The authors give a simple proof of the following theorem of Kulikov [Trudy Moskov. Mat. Obšč. **1** (1952), 247–326; MR **14**, 132; p. 254]. Any countably generated torsionfree (unitary) Z_p-module containing no elements of infinite height is a direct sum of cyclic Z_p-modules.

Moreover, a countably generated mixed Z_p-module without elements of infinite height is constructed, the torsion-submodule of which is not a direct summand. This example shows that in formulating the analogue for mixed modules of the above theorem by Kulikov, the requirement that the torsion part of the module should be a direct summand cannot be omitted. *Authors' summary*

2694:

Erdélyi, Mária. On direct summands of abelian torsion groups. Acta Univ. Debrecen. **2** (1955), 145–149. (Hungarian. English summary)

Generalizing a theorem of Kaplansky [*Infinite abelian groups*, Univ. of Michigan Press, Ann Arbor, 1954; MR **16**, 444; p. 25], we prove the following result: An element a of an abelian p-group G is contained in a finite direct summand of G if and only if all the elements of the cyclic group generated by a have finite height in G. The proof is based on a well-known theorem of Kulikov [Mat. Sb. (N.S.) **9 (51)** (1941), 165–181; MR **2**, 308], which says that a pure subgroup S of G is a direct summand if the orders of elements contained in S form a bounded set. Our result naturally gives a criterion of being contained in a finite direct summand even for elements of an arbitrary abelian torsion group. *Author's summary*

2695:

Murnaghan, F. D. The orthogonal and symplectic groups. Comm. Dublin Inst. Adv. Studies. Ser. A, no. 13 (1958), 146 pp. 12s. 6d.

This series of twenty lectures, given in Dublin in 1957, is principally an exposition of representations of the unitary, orthogonal and symplectic groups by linear transformations, and the presentations of these representations by matrices. Defining a vector as a pair (X, c), where X is a non-singular $n\times n$ matrix and c an $n\times 1$ matrix, and defining a vector space as the set of all such pairs for a fixed basis matrix X, the author uses the equation $Xc=X'c'$ to relate the change of basis from X to X' to the change of coordinates from c to c', and shows that similar matrices B and ABA^{-1} present the same linear transformation relative to the bases X and X' if $X'A=X$. Transformations that preserve the first and second scalar products C^tC or $C^{\dagger}C$ are called unitary or orthogonal, while those that preserve C^tMC for certain matrices M are called Lorentz or symplectic transformations. Canonical factorizations for matrices of the unitary and orthogonal groups are given that display the r angular parameters, certain ones of which characterize the class. Lecture five discusses the adjoint representation in the parameter space. Assuming that the $m\times m$ matrix Y corresponds to the point y in the parametric r-space, and that the characteristic matrices M_j are the values of

$\partial Y/\partial y_j$ at $Y=E$, then the $m\times m$ matrix A corresponds to the $r\times r$ matrix $A'=(\alpha_{ij}')$ in the adjoint representation if the similarity mapping $Y\to AYA^{-1}$ induces the linear mapping $M_j\to\sum_i M_i\alpha_{ij}'$. Volume elements and integrals are computed over a compact group. After discussing reducibility, Kronecker products, and orthogonality relations, the irreducible constituents $\{\lambda\}$ of the Kronecker mth power are derived, one for each partition (λ) of m. These, multiplied by powers of the one-dimensional determinant representation, furnish all continuous irreducible representations of the n-dimensional unitary group. In similar manner lectures 12 to 14 treat the irreducible representations for rotation groups and orthogonal groups, and lectures 15 and 16 treat those of the U-symplectic and O-symplectic groups. The last four lectures are concerned with the analysis of product representations and certain compounded representations of unitary, orthogonal, and U-symplectic groups. For example, rules are given for expanding certain iterated representation functions such as $\{2\}\otimes\{1^5\}=\{61^4\}+\{531^2\}+\{42^2\}$.

J. S. Frame (East Lansing, Mich.)

2696:

Deskins, W. E. On the radical of a group algebra. Pacific J. Math. 8 (1958), 693–697.

Let $\mathfrak{R}$ be the radical of the group algebra $\mathfrak{A}(G)$ of a finite group over a field of characteristic p. Since $\mathfrak{R}$ is non-zero if and only if p divides the order of G, it may be expected that $\mathfrak{R}$ is related to the Sylow p-groups of G. Thus let $\mathfrak{R}'$ be the intersection of all the left ideals of $\mathfrak{A}(G)$ generated by the radicals of the group algebras of the Sylow p-groups of G. Whilst $\mathfrak{R}$ and $\mathfrak{R}'$ are not always equal, the author proves that $\mathfrak{R}=\mathfrak{R}'$ holds in the following cases: (A) G has a normal subgroup H such that G/H is isomorphic with a Sylow p-group of G; (B) G is supersoluble; (C) G is soluble and its order is divisible by p but not by p^2. The paper also contains a contribution to the problem of establishing the existence of an invariant p-subgroup under suitable conditions, which include the assumption that $\mathfrak{A}(G)$ should be bound to $\mathfrak{R}$ in the sense of M. Hall [Trans. Amer. Math. Soc. **48** (1940), 391–404; MR **2**, 122].

W. Ledermann (Manchester)

2697:

Farahat, H. K.; and Higman, G. The centres of symmetric group rings. Proc. Roy. Soc. London Ser. A **250** (1959), 212–221.

The center K_n of the integral group ring of the symmetric group S_n is shown to be generated by the n elements $Z_i=\sum_{t(\alpha)=i}\alpha$ $(i=1, 2, \cdots, n)$, where $t(\alpha)$ denotes the number of cycles (including 1-cycles) in an element α in S_n. In the proof a ring K is constructed such that every K_n $(n=1, 2, \cdots)$ is a homomorphic image of K, and K has the conjugate classes in the restricted symmetric group S on the positive integers $(S=\bigcup S_n)$ as basis over the ring of polynomials $f(x)$ for which $f(n)$ is integral whenever n is an integer. From the result, in combination with Farahat's [Proc. London Math. Soc. (3) **6** (1956), 501–517; MR **19**, 634] result, follows the p-core (equivalently, p-residue) criterion for two representations to belong to the same p-block [Nakayama, Jap. J. Math. **17** (1941), 411–423; MR **3**, 196; R. Brauer, Trans. Roy. Soc. Canada Sect. III (3) **41** (1947), 11–19; MR **10**, 678; G. de B. Robinson, ibid. 20–25; MR **10**, 678].

T. Nakayama (Nagoya)

2698:

Šutov, È. G. Potential conjugacy of elements in semigroups. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 105–119. (Russian)

Two elements x, y of a semigroup A (with unit element e) are conjugate in A if there exist elements $t, t'\in A$ such that $tt'=t't=e$, $x=tyt'$. Potentially conjugate elements in A require, of course, the existence of a semigroup $A'\supseteq A$ (with the same unit element) such that x and y are conjugate in A'.

The author determines simple consequences of this concept of which the following application is typical: For any group G, there exists a covering group G' in which all elements of the same order are conjugate.

L. J. Paige (Los Angeles, Calif.)

2699:

Kolibiarová, Blanka. Über kommutative periodische Halbgruppen. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 127–135. (Slovak. Russian and German summaries)

Let S be a commutative semigroup in which $\{x^n\}_{n=1}^{\infty}$ is finite for all $x\in S$, and let $I(S)$ be the set of idempotent elements of S. For $e\in I(S)$, let $K(e)$ be the set of all $x\in S$ such that $x^n=e$ for some positive integer n. The mapping $x\to e$, where $x\in K(e)$, is a homomorphism of S onto $I(S)$. For a generalization of this result, see Hewitt and Zuckerman [Trans. Amer. Math. Soc. **83** (1956), 70–97; MR **18**, 465; Theorem 4.13].

For $x\in S$, write $(x)=x\cup Sx$: this is the smallest ideal in S containing x, and is called principal. For $x\in S$, let F_x be the set of all $y\in S$ such that $(y)=(x)$. If $x'\in F_x$ and $y'\in F_y$, then $x'y'\in F_{xy}$. Thus, with $F_x\circ F_y=F_{xy}$, the set $\mathfrak{F}$ of all F_x forms a semigroup that is a homomorphic image of S, under the mapping $x\to F_x$. Partially order $\mathfrak{F}$ by the definition $F_x\leq F_y$ if $(x)\subset(y)$. Sample theorems: Suppose that $F_x\leq F_y$. Then $F_{xy}=F_y$ if and only if F_y is a group. If every properly descending chain $(x_1)\supset(x_2)\supset\cdots$ in S is finite, then every descending chain of idempotents in S is also finite ($e\leq f$ means $ef=e$). S has a minimal principal ideal N if and only if $I(S)$ has a zero f, and then N is the maximal group containing f.

E. Hewitt (Seattle, Wash.)

2700:

Parízek, Bohumír; and Schwarz, Štefan. On the multiplicative semigroup of residue classes (mod m). Mat.-Fyz. Časopis. Slovensk. Akad. Vied **8** (1958), 136–150. (Slovak. Russian and English summaries)

Let m be a positive integer with prime factorization $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ ($\alpha_1, \cdots, \alpha_r$ positive integers). Let S_m be the multiplicative semigroup of all residue classes $[x]$ modulo m, and let G_1 be the subgroup of S_m consisting of all $[a]$ such that $(a, m)=1$. It is easy to see that if T is any semigroup with unit and H is any subgroup of T containing the unit, then $T=\bigcup_{x\in T}xH$, where the sets xH are either disjoint or identical. Applying this decomposition to S_m and G_1, the authors first prove that the sets $[x]G_1$ all have the form $[p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}]G_1$, $0\leq k_i\leq\alpha_i$ $(i=1, 2, \cdots, r)$, and that the sets just described are pairwise disjoint. The set $[p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}]G_1$ consists of

exactly $\varphi(p_1^{\alpha_1-k_1}p_2^{\alpha_2-k_2}\cdots p_r^{\alpha_r-k_r})$ elements. The semigroup S_m contains exactly 2^r idempotents. Each idempotent can be written as $[p_1^{l_1}p_2^{l_2}\cdots p_r^{l_r}a]$, where $a \in G_1$ and $l_i = 0$ or α_i $(i = 1, 2, \cdots, r)$. The set of all idempotents in S_m is a Boolean algebra with the usual ordering ($e \leq f$ if $ef = e$). A number of other properties of S_m are obtained.

E. Hewitt (Seattle, Wash.)

2701:

Kajan, Juraj. Remark on the structure of a type of left simple semigroups. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 8 (1958), 187–192. (Slovak. Russian and English summaries)

Let S be a semigroup in which $Sa = S$ for all $a \in S$ and $ax = ay$ implies $x = y$ for all $a, x, y \in S$. Suppose also that S is not a group (S is a group if and only if it contains an idempotent). For $a, b \in S$, let $S_a{}^b = \{x : x \in S, xb = a\}$. Let $R_a = (\bigcup_{b \in S} S_a{}^b)'$. Then $R_a \cap \{a\}'$ is the largest right ideal of S not containing a. Let $x_a{}^b$ be any element of $S_a{}^b$. Then $S = R_a \cup (\bigcup_{b \in S} S_a{}^a x_a{}^b)$, and the sets written on the right of this equality are pairwise disjoint.

E. Hewitt (Seattle, Wash.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 2877, 3024, 3025, 3112.

2702:

★**Cohn, P. M. Lie groups.** Cambridge Tracts in Mathematics and Mathematical Physics, no. 46. Cambridge University Press, New York, N.Y., 1957. vii + 164 pp. $4.00.

This is a concise and elementary textbook on the beginnings of the theory of Lie groups, covering the so-called Lie's fundamental theorems. It also includes the theory of universal covering groups. The readers are required to have only elementary knowledge about groups, vector spaces, topology, and analysis. Numbers of examples are inserted to help the readers.

Chapter 1 introduces the elementary notions in the theory of (real) analytic manifolds. Chapter 2 introduces the definitions of topological groups, Lie groups, local Lie groups, analytic subgroups of Lie groups, followed by discussions on one-parameter subgroups.

Chapter 3 treats algebras of infinitesimal right translations, that is, infinitesimal transformations which are left invariant, of Lie groups and of analytic subgroups, in particular one-parameter subgroups. These are also generalized to the case of Lie groups of transformations. Chapter 4 treats differential forms, exterior differentiation, their behavior under analytic mappings, Maurer-Cartan forms (or left-invariant forms), and their relations with infinitesimal right translations.

Chapter 5 is concerned with Lie's fundamental theorems. They are formulated roughly in the following fashion. The first and its converse: The multiplication functions of local Lie groups are characterized by the fact that they satisfy a certain type of partial differential equation. The second and its converse: The necessary and sufficient conditions for a set of infinitesimal transformations to be the algebra of infinitesimal translations of a local Lie group. The third and its converse: The abstract algebraic characterization of algebras of infinitesimal translations of local Lie groups and the determination up to local isomorphism of a local Lie group by its algebra.

The main subjects discussed in Chapter 6 are the following: The canonical charts; the fact that continuous homomorphisms of Lie groups are analytic; the correspondence between analytic subgroups of a Lie group and subalgebras of its Lie algebra; the fact that the closed subgroups of Lie groups are analytic; homomorphism, kernel and quotient of Lie groups; homomorphism of Lie algebras and their relations with homomorphism of Lie groups; the adjoint representation of a Lie group.

The final chapter is about the universal covering group. The book closes with an appendix, in which a theorem on the integration of a system of total differential equations used in Chapter 4 is proved.

M. Kuranishi (Nagoya)

2703:

Hochschild, G.; and Mostow, G. D. Extensions of representations of Lie groups and Lie algebras. I. Amer. J. Math. **79** (1957), 924–942.

Let K be a connected Lie group and let G be a closed connected normal subgroup of K. Let ρ be a representation of G on a vector space V. (All representations here are finite dimensional.) The question considered is: when does there exist a representation σ of K whose restriction to G contains ρ as a subrepresentation? The following theorem is the main result. Assume that there exists an analytic subgroup H of K such that $K = HG$, $H \cap G$ is compact, and there exists a representation of H that is faithful on $H \cap G$. Let ρ' be the associated semisimple representation to ρ. (To any representation there is associated, uniquely up to equivalence, a completely reducible representation called the associated semisimple representation.) Assume further that $\rho'(xyx^{-1}y^{-1}) = 1$ for every $y \in K$ and every x in the radical of G. Then σ exists. Furthermore if σ' is the associated semisimple representation to σ then σ can be chosen so that the kernel of σ' contains the kernel of ρ'. Note: the representation space for σ is constructed using representative functions.

The theorem above together with other results are shown to simplify the proofs of basic theorems on faithful representations due to E. Cartan, Goto and Mal'cev. Also given is a simple proof of a theorem of Zassenhaus on the extendability of representations of Lie algebras. (This result is used in Zassenhaus' proof of Ado's theorem.) Finally the results are related to representative functions.

B. Kostant (Berkeley, Calif.)

2704:

Mostow, G. D. Extension of representations of Lie groups. II. Amer. J. Math. **80** (1958), 331–347.

This paper is a sequel to part I, reviewed above. The question involved, as in part I, is to extend a representation ρ from a closed, connected, normal subgroup G of a connected Lie group L to a representation of L. In part I a condition for doing this was given. One of the assumptions there was that modulo a compact group, L is a splittable extension of G. Here no assumptions are made and a necessary and sufficient condition is given. The method uses those of part I and a new technique which relies on the theory of algebraic groups. The latter is needed to deal with topological difficulties. The main theorem asserts

the following. In the notation above let ρ' denote the associated semisimple representation of ρ. Let N be the radical of the commutator group of L. Let P be the intersection of the kernels of all finite dimensional representations of L. Then the following conditions for the extendability of ρ are necessary and sufficient: (1) The representation ρ is trivial on $P \cap G$; (2) ρ' is trivial on $G \cap N$; and (3) if σ is the representation on GN defined by $\sigma(gu) = \rho'(g)$, where $g \in G$, $u \in N$ (the existence of σ being assured by (2)), then σ is continuous in the induced topology of GN. *B. Kostant* (Berkeley, Calif.)

2705:

Dixmier, J. Sur les représentations unitaires des groupes de Lie nilpotents. I. Amer. J. Math. **81** (1959), 160–170.

Let G be a real connected nilpotent Lie group and H a closed connected abelian normal subgroup of G. The author shows that H must then be "regularly imbedded" in G in the sense of Mackey [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 537–545; MR **11**, 158]. On the basis of this he obtains the result that any unitary factor representation of G is unitarily equivalent to the representation of G induced by some representation by scalar operators of a closed connected subgroup of G that contains H. It then follows from an induction argument that every unitary representation of G is of type I.

K. deLeeuw (Stanford, Calif.)

2706:

Nagano, Tadashi. Compact homogeneous spaces and the first Betti number. J. Math. Soc. Japan **11** (1959), 4–9.

The author proves among others: Let G be a compact connected Lie group G and let $M = G/H$ be an n-dimensional homogeneous space (H itself is not necessarily connected). If B_1 is the first Betti number of M, then $B_1 = n - \dim S(p)$, where $S(p)$ is the orbit of an arbitrary point p in M under the maximal (connected) subgroup S of G.

{The reasonings in the proof and several among the corollaries were already known.}

S. Bochner (Princeton, N.J.)

2707:

Oniščik, A. L. Transitive groups of transformations of compact homogeneous spaces. Dokl. Akad. Nauk SSSR **124** (1959), 520–523. (Russian)

Results (all groups involved are supposed to be connected): Let G be a compact simple Lie group acting transitively on $X = G/H$. Suppose H is a so-called R-subgroup [E. B. Dynkin, Mat. Sb. (N.S.) **30** (**72**) (1952), 349–462; MR **13**, 904], and G' a subgroup of G transitive on X. The possibilities for G, H, G' are listed.

Suppose a compact Lie group G acts transitively on X. If G contains a simple subgroup that is transitive on X, then G is simple.

If G is a simple complex Lie group acting transitively and analytically on a complex compact manifold X with $\chi(X) > 0$, then the maximal connected group $A(X)$ of analytic homeomorphisms of X has trivial centre. $A(X)$ is locally isomorphic with G up to a number of explicitly listed exceptions. *W. T. van Est* (Leiden)

2708:

Poncet, Jean. Groupes de Lie compacts de transformations de l'espace euclidien et les sphères comme espaces homogènes. Comment. Math. Helv. **33** (1959), 109–120.

It is proved that if a compact connected Lie group G acts on E^n such that all orbits are $(n-1)$-spheres and a single fixpoint, then G acts linearly on E^n, i.e. E^n may be coordinatized such that the action of G is linear. Earlier results of Montgomery and Zippin and a result of Borel on homogeneous homology spheres combined with the above statement imply that a compact connected group acts linearly on E^n as soon as there is at least one $(n-1)$-dimensional orbit.

In proving the above result it is first shown that a compact connected Lie group acting effectively on S^n is a rotation group. Then assuming that G acts effectively on E^n with spherical $(n-1)$-dimensional orbits and a single fixpoint, a fixed spherical orbit admits an equivariant imbedding in E^n with G acting linearly. This imbedding is extended to a 1-1 map from E^n with the given action of G onto E^n with a linear action of G. By invariance of domain it follows that the map is an equivalence.

By similar arguments it follows that if G acts effectively on E^n such that the orbits are $(n-2)$-spheres and fixpoints then again the action of G is linear. This implies a result announced before [C. R. Acad. Sci. Paris **245** (1957), 13–15; MR **19**, 753]. Some further remarks on the occurrence of $(n-2)$-dimensional orbits are added.

W. T. van Est (Leiden)

2709:

Abe, Eiichi. On the groups of C. Chevalley. J. Math. Soc. Japan **11** (1959), 15–41.

For a field K and a simple Lie algebra $\mathfrak{g}$ over the complex number field, Chevalley [Tohoku Math. J. **7** (1955), 14–66; MR **17**, 457] defined a Lie algebra $\mathfrak{g}_K$ and a group G_K of automorphisms of $\mathfrak{g}_K$. In the present paper G_K is considered as an algebraic group under the assumption that K is infinite [cf. Ono, J. Math. Soc. Japan **10** (1958), 307–313; MR **20** #7076]. The following is proved. G_K is an irreducible simple algebraic group whose Lie algebra is isomorphic to $\mathfrak{g}_K$, provided that the characteristic of K satisfies certain conditions. If the characteristic of K is not 2, then any biregular and birational automorphism of G_K is inner except when $\mathfrak{g}$ is of type (D_4). For the latter case the outer automorphism group of G_K is of order 3. The author also identifies the commutator group of G_K with classical groups when $\mathfrak{g}$ is of one of the types (A_l), (B_l), (C_l), (D_l) by a method different from the reviewer's [Trans. Amer. Math. Soc. **84** (1957), 392–400; MR **19**, 247].

R. Ree (New York, N.Y.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

See also 2905.

2710:

Cohen, Haskell; and Krule, I. S. Continuous homomorphic images of real clans with zero. Proc. Amer. Math. Soc. **10** (1959), 106–109.

A real clan S is a topological semigroup with zero and unit whose underlying space is a closed interval of real

numbers. The authors show that a continuous homomorphic image of S is either (1) a real clan or (2) a triad (in which case its zero is an endpoint). An example is given to illustrate (2). *R. J. Koch* (Baton Rouge, La.)

FUNCTIONS OF REAL VARIABLES

See also 2719, 2760, 2839.

2711:

Mukhoti, S. R. On a generalisation of some theorems of the differential calculus. Bull. Calcutta Math. Soc. **50** (1958), 87–93.

Some elementary mean value theorems and their usual proofs are given in terms of (upper, lower) semi-continuous functions and their (upper, lower) (left, right) derivatives. *L. A. Rubel* (Urbana, Ill.)

2712:

Sengupta, H. M.; and Ganguli, P. L. On a class of steadily increasing functions with an everywhere dense set of points of discontinuity. Bull. Calcutta Math. Soc. **50** (1958), 9–18.

Let $P=\{p_n: 1\leq n<\infty\}$ be a countable set of points in the open unit interval (0, 1), and let $A=\{a_n: 1\leq n<\infty\}$ be a countable set of positive real numbers such that $\sum_n a_n=1$. Let $p_0=\inf p_n$. For $0\leq x\leq 1$, define $f(x)=0$ if $x<p_0$, and define $f(p_0)=0$ if $p_0\notin P$; for all other x define $f(x)=\sum\{a_n: p_n\leq x\}$. Then $f(0)=0$, $f(1)=1$, f is a non-decreasing function which is continuous from the right at each point in [0, 1], and f is discontinuous at each point p_n and at no other points. Moreover, the saltus of f at p_n is a_n. A necessary and sufficient condition that the function f so defined be strictly increasing is that the set P be dense in [0, 1]. Every right-continuous non-decreasing f on [0, 1], whose discontinuities are dense, and for which $f(0)=0$, $f(1)=1$, arises this way. The paper under review proves parts of these statements for the special case in which P is the set of dyadic rationals, provided the set A is paired with the set P in such a way that the series $\sum_n a_n$ satisfies a restrictive condition. The method of proof used is the expansion of reals in dyadic notation.

F. B. Wright (New Orleans, La.)

2713:

Kennedy, P. B. A remark on continuity conditions. Proc. Amer. Math. Soc. **10** (1959), 203–204.

Let E be a set in the interval (0, 1), $f(x)$ be defined in (0, 1), and $\omega(t)$ be a positive increasing function such that $\omega(t)\to 0$ as $t\to 0$. $\omega(t)$ is said to be a modulus of continuity of $f(x)$ in E if $|f(x+h)-f(x)|\leq\omega(|h|)$ for $x\in E$ and all $x+h$ in (0, 1). The author proves the following theorem. For any set E in (0, 1), there is a function $f(x)$ such that $\omega(t)$ is a modulus of continuity of $f(x)$ in E and $f(x)$ is discontinuous at every interior point of the complement of E.

As a special case, it follows that there is a function belonging to the Lipschitz class in a set E but not in any interval. This was proved by the author [J. London Math. Soc. **33** (1958), 196–207; MR **20** #1735] as a corollary of a deep theorem. *S. Izumi* (Sapporo)

2714:

Shukla, U. K.; and Garg, K. M. A note on a function of non-symmetrical differentiability. Ganita **9** (1958), 27–32.

In an earlier paper [Ganita **8** (1957), 81–104; MR **20** #6497] the first named author purported to prove that a certain real-valued function f is, among other things, continuous throughout the unit interval I. The present paper appears to establish, in contradiction with this earlier result, that this function f has points of discontinuity and is indeed, it is now said to be possible to show, discontinuous at the points of a set dense in I.

T. A. Botts (Charlottesville, Va.)

2715:

Waterman, Daniel. On an integral of Marcinkiewicz. Trans. Amer. Math. Soc. **91** (1959), 129–138.

The author proves a result analogous to certain inequalities of J. Marcinkiewicz [Ann. Soc. Polon. Math. **17** (1938), 42–50]: If $F(t)$ is the primitive of a function f of class $L^p(-\infty, \infty)$ $(p>1)$, he defines $\mu(\tau)=\{\int_0^\infty[F(\tau+t)+F(\tau-t)-2F(\tau)]^2/t^2dt\}^{1/2}$ and proves that there exist constants A_p, B_p such that

$$A_p\|f\|_p\leq\|\mu\|_p\leq B_p\|f\|_p.$$

His methods are akin to those used by Zygmund to prove Marcinkiewicz's inequalities [A. Zygmund, Trans. Amer. Math. Soc. **55** (1944), 170–204; MR **5**, 230].

U. S. Haslam-Jones (Oxford)

2716:

Young, G. S. Extensions of Liouville's theorem to n dimensions. Math. Scand. **6** (1958), 289–292.

Let f be a continuous vector map from the closure of a domain D in Euclidean n-space E^n into E^n which is of class C' in D and has a non-negative Jacobian $J(f)$ in D. Suppose there is a spherical region of radius r whose closure is in D on which f is one-to-one and $J(f)$ is not less than a positive constant k. It is shown that the oscillation of f on the boundary of D is not less than $2k^{1/n}r$. As a corollary it is proved that if f is a vector map from E^n into E^n of class C' having a non-negative Jacobian $J(f)$ and a limit at infinity then $J(f)$ is identically zero. For analytic functions in the plane this result yields Liouville's theorem.

P. V. Reichelderfer (Columbus, Ohio)

2717:

Ostrowski, Alexandre. Un critère d'univalence des transformations dans un R^n. C. R. Acad. Sci. Paris **247** (1958), 172–175.

For the purpose of improving a proof of the implicit function theorem, the author independently gives a method which is essentially the same as the one the reviewer published before [Amer. Math. Monthly **64** (1957), 725–726; MR **20** #3243; see p. 726]. The author shows that the same proof also works for the case of complex variables. *H. Yamabe* (Osaka)

2718:

Ostrowski, Alexandre. Un nouveau critère d'univalence des transformations dans un R^n. C. R. Acad. Sci. Paris **248** (1959), 348–350.

This note is a continuation of the one reviewed above. Let $\Delta(x)$ be the Jacobian matrix at x of a transformation

$F(x)$ defined over a convex n-dimensional domain, A be a fixed non-singular matrix, and E be the unit matrix. All the matrices are considered as linear transformations over an n-dimensional vector space. The norms of these linear transformations are easily defined if a norm is introduced in the vector space. Then (i) $\|\Delta(x)A^{-1}-E\| \leq 1$ and $\|\Delta(x)-A\| \leq (\|A^{-1}\|)^{-1}$ are sufficient for $F(x)$ to be a homeomorphism; but (ii) $\|\Delta(x)-A\| \leq \|A\|$ is not in general sufficient, and this is illustrated by an example. When the domain is not convex $\|\Delta(x)A^{-1}-E\| \leq J_D$ is a sufficient condition, with $J_D = \sup_{P_1,P_2 \in D} |P_1P_2|_D/|P_1P_2|$, where $|P_1P_2|_D$ is understood as the distance between two points inside D. When D is convex, $J_D = 1$.

H. Yamabe (Minneapolis, Minn.)

MEASURE AND INTEGRATION

See also 3042.

2719:

Marcus, Solomon. Sur un théorème de F. B. Jones. Sur un théorème de S. Kurepa. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 433–434.

The reviewer showed in 1942 that there exists a Hamel basis which contains a perfect set. The author shows how this theorem may be quickly deduced from one of von Neumann's theorems [dealing with algebraically independent number sets]. Actually the first proof of this theorem (as far as the reviewer now knows) is due to Mark Kormes [Bull. Amer. Math. Soc. **32** (1926), 689–693]. Kormes' argument is of the same type as given by the author but somewhat simpler.

The following theorem is due to S. Kurepa [Glasnik Mat.-Fiz.-Astr. Ser. II **2** (1956), 89–93; MR **19**, 408]: A convex function f from the real numbers to the real numbers must be continuous if f is bounded over a set E such that $E+E=\{x+y: x \in E \text{ and } y \in E\}$ has positive interior measure. The author shows that this theorem is an easy consequence of a previously known theorem of this type [when E is an interval] and one of Sophie Piccard's [cf. Kormes, loc. cit.] theorems.

F. B. Jones (Chapel Hill, N.C.)

2720:

Taylor, S. J. On strengthening the Lebesgue Density Theorem. Fund. Math. **46** (1959), 305–315.

Solving a problem posed by S. Ulam in 1937 in the *Scottish Book* [Lwów, 1935–1941; translated by S. Ulam, Los Alamos, 1957], the author establishes the following theorem. Given a Lebesgue measurable set E on the real line, there exists a continuous, monotone increasing, real-valued function ψ on the set of positive reals such that $\lim_{x \to 0+} \psi(x) = 0$ and such that (†) for all ξ in E except for a set of Lebesgue measure zero,

$$\lim_{\xi \in I, |I| \to 0} \frac{|I \cap CE|}{|I|\psi(|I|)} = 0,$$

where CE denotes the complement of E, I denotes a closed interval, and $|A|$ denotes the Lebesgue measure of the measurable set A. It is also shown that no such function ψ exists for which condition (†) holds uniformly for all measurable sets E on the line. Several other related theorems are proved, some of them extending work of Besicovitch [J. London Math. Soc. **31** (1956), 48–53; **32** (1957), 170–178; MR **18**, 24; **19**, 639] on linear density. Analogous results in n dimensions are noted briefly, and in this connection an unsolved problem is posed.

T. A. Botts (Charlottesville, Va.)

2721:

Marczewski, E.; and Steinhaus, H. On a certain distance of sets and the corresponding distance of functions. Colloq. Math. **6** (1958), 319–327.

For an arbitrary σ-finite measure space (X, M, μ) let M_0 be the class of all sets of finite μ-value and $\mathscr{L}_\mu$ the family of all μ-integrable functions. The authors study the functions $\sigma(A, B) = \mu(A \Delta B)/\mu(A \cup B)$ on $M_0 \times M_0$ and

$$\sigma(f, g) = \frac{\int |f(x)-g(x)| d\mu(x)}{\int \max(|f(x)|, |g(x)|, |f(x)-g(x)|) d\mu(x)}$$

on $\mathscr{L}_\mu \times \mathscr{L}_\mu$ (defined as 0 whenever their denominators vanish). These functions are shown to share various important properties with the familiar pseudodistances represented by their numerators. In particular, they themselves are pseudodistances which make the sets M_0 and $\mathscr{L}_\mu$ into complete pseudometric spaces, respectively. Moreover, they are shown to have practical significance as measures of the qualitative or quantitative discrepancy between 2 biotopes; a forthcoming paper by the authors is devoted to this application.

H. M. Schaerf (St. Louis, Mo.)

2722:

Poprużenko, J. Sur certaines représentations des fonctions d'ensemble à variation bornée. II. (Fonctions σ-normales.) Colloq. Math. **5** (1958), 176–184.

Cette note continue une publication antérieure de l'auteur [même Colloq. **5** (1957), 43–50; MR **20** #1744], citée [I], dont on conserve les notations. F: fonction réelle, définie sur le σ-corps $\mathfrak{M}$, de variation totale bornée. $V_F(E)$: variation totale de F sur $E \in \mathfrak{M}$. F est dite "σ-normale" si, pour toute famille dénombrable (E_n) d'ensembles de $\mathfrak{M}$ deux à deux disjoints, $|F(\sum E_n)| \leq \sum |F(E_n)|$. Théorème I.—Hypothèses: $P(\mathfrak{m})$ est vérifiée, $\mathfrak{M}$ = famille des sous-ensembles de $\mathscr{M}$, F est σ-normal. Conclusions: (A) Si F s'annule sur les ensembles uniponctuels, F s'annule identiquement. (B) Si, pour un élément x_0 de $\mathscr{M}$, $F((x_0))$ est non nul, il existe un ensemble H_0 dénombrable et $E_0 \subset H_0$ tels que $|F(E)| \leq |F(E \cdot H_0)|$ et $F(E_0) \neq 0$. Théorème II.—Hypothèses: $\mathscr{M}$ est un ensemble fondamental de nombres irrationnels [cf. Fund. Math. **42** (1955), 319–338; MR **17**, 952], $\mathfrak{M}$ = famille des boréliens de $\mathscr{M}$, F est σ-normale. Conclusions: (A) Si F s'annule sur tout ensemble de $\mathfrak{M}$ de puissance $< \mathfrak{m}$, F s'annule identiquement. (B) Si, pour un ensemble X_0 de $\mathfrak{M}$ de puissance $< \mathfrak{m}$, $F(X_0)$ est non nul, il existe dans $\mathfrak{M}$ un ensemble H_0 de puissance $< \mathfrak{m}$ et un sous-ensemble E_0 de H_0 tels que $|F(E)| \leq |F(E \cdot H_0)|$ et $F(E_0) \neq 0$. Les démonstrations utilisent les techniques de [I]. {Remarques du rapporteur—(i) Dans la formulation de Lemme I, les relations entre F, χ et ϕ ont été omises. (ii) Observant que, pour une F σ-normale, V_F est σ-additive, il suffit d'établir Th. I et II pour une F σ-additive, ce qui abrège les démonstrations, ainsi Th. I(A) devient corollaire d'un théorème de S. Ulam, ibid. **16** (1930), 140–150.}

Chr. Pauc (Nantes)

2723:

Weston, J. D. A counter-example concerning Egoroff's theorem. J. London Math. Soc. **34** (1959), 139–140.

In connection with extensions of Egoroff's theorem from sequences to families of functions indexed by a parameter varying over a real interval, there is constructed (using the axiom of choice) an example of a family of functions f_h on the half-open interval $X=[0, 1)$ such that (i) for each h in X the function f_h vanishes except at a single point where its value is 1, (ii) for each x in X, $f_h(x)\to 0$ as $h\to 0$, and (iii) this convergence is uniform on no set of positive measure.

T. A. Botts (Charlottesville, Va.)

2724:

Fleming, W. H. Nondegenerate surfaces of finite topological type. Trans. Amer. Math. Soc. **90** (1959), 323–335.

The author extends Morrey's theorem to a nondegenerate Fréchet surface of finite area of any finite topological type, by establishing, for such a surface, the existence of a quasi-conformal representation on a corresponding parallel slit domain. In addition to fundamental notions, introduced by T. Rado, J. W. T. Youngs, and others, and to machinery taken from the classical researches of Courant and Shiffman, the paper uses the reviewer's ε, δ-gratings and Cesari's fine cyclic elements. It is remarkable that so many years should have had to elapse before the extension of Morrey's theorem was discovered.

L. C. Young (Madison, Wis.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 2611, 2851, 2879, 2892.

2725:

Schoenfeld, Lowell. Some remarks on the logarithmic function in the complex plane. Math. Mag. **32** (1958/59), 189–202.

Believing that standard texts on elementary complex variable theory do not furnish in sufficient detail the development of the logarithm function and its properties, the author presents a careful treatment to fill this gap.

M. S. Robertson (New Brunswick, N.J.)

2726:

★Riabouchinsky, Dimitri P. Recours aux calculs numériques et aux interprétations graphiques pour la mise en évidence des particularités de la théorie des variables complexes généralisées. Actes du colloque de calcul numérique, Périgueux, 1957, pp. 43–44. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 80, Paris, 1959. vii+87 pp. 1800 francs.

2727:

Derwidué, L. A propos d'un théorème de H. Bückner. Mém. Publ. Soc. Sci. Arts Lett. Hainaut **71** (1957), no. 2, 55–64.

This paper is concerned with a theorem stating that a Hurwitz polynomial can be written as a determinant in tri-diagonal form, and the coefficients in this determinant are expressed in terms of Hurwitz determinants. The theorem is attributed to H. Bückner [Quart. Appl. Math. **10** (1952), 205–213; MR **14**, 145]. It is actually a special case (concerned here only with real coefficients) of a theorem of E. Frank [Amer. Math. Monthly **59** (1952), 300–309; MR **14**, 6]. This theorem and the expressions for the coefficients are also found in Frank, Bull. Amer. Math. Soc. **52** (1946), 144–157, 890–898 [MR **7**, 295; **8**, 154], where they form the basis of the stability criterion developed there (cf. p. 152) for characteristic polynomial equations with real or complex coefficients.

E. Frank (Chicago, Ill.)

2728:

Frank, Evelyn. A note on certain determinantal equations. Math. Nachr. **19** (1958), 182–185.

Applying previous results due to Frank [Amer. Math. Monthly **59** (1952), 300–309; MR **14**, 6] and Grommer to continued fraction expansions, the author obtains a few theorems concerning the nature of the roots of some determinantal equations. E.g., let $P(z)=(a_{ij})$, where $a_{11}=f_1z+g_1+1$, $a_{ii}=f_iz+g_i$ $(i=2, \cdots, n)$, $a_{ij}=e_i$ for $i=j+1$ $(j=1, 2, \cdots, n-1)$, $a_{ij}=-1$ for $i=j-1$ $(j=2, 3, \cdots, n)$, and all remaining elements which are not on the three main diagonals are zero. If f_i are real and g_i pure imaginary or zero, and if k of the f_i are negative and $(n-k)$ are positive, the equation $P(z)=0$ has k roots with positive real parts and $(n-k)$ roots with negative real parts.

J. W. Andrushkiw (Newark, N.J.)

2729:

Vicente Gonçalves, J. Quelques limites pour les modules des zéros d'un polynome. Univ. Lisboa. Revista Fac. Ci. A (2) **6** (1957/58), 83–120.

This article is concerned with giving new derivations, generalizing and comparing various known bounds on all the zeros of a polynomial in a complex variable [see Chapter VII in M. Marden, *Geometry of the zeros of a polynomial in a complex variable*, Amer. Math. Soc., New York, 1949; MR **11**, 101]. An approximating sequence is constructed for one of Cauchy's bounds. For the bounds due to Westerfield and Kojima conditions for either one of these to be better than the other are found. The inequality of Fujiwara for all the zeros is generalized to one for p zeros of an nth degree polynomial $(p\leq n)$.

M. Marden (Milwaukee, Wis.)

2730:

Parodi, Maurice. Sur deux équations trinomes. C. R. Acad. Sci. Paris **248** (1959), 171–172.

When $|a|>1+|b|$, the polynomial $f(z)=z^n+az+b$ is known to have just one zero Z with $|Z|<1$. Starting with the formula $Z=(1/2\pi i)\int_C [zf'(z)/f(z)]dz$, where C is the unit circle, the author expresses Z in an infinite series involving powers of a and b.

M. Marden (Milwaukee, Wis.)

2731:

Parodi, Maurice. Sur la localisation des racines des équations réciproques. C. R. Acad. Sci. Paris **248** (1959), 902–904.

By use of the substitution $z=x+(1/x)$, the equation

$$x^{2n}+a_1x^{2n-1}+\cdots+a_{2n-1}x+a_{2n}=0,$$

where $a_{2n-k}=a_k$ for $k=0, 1, \cdots, n$, is transformed into the determinant

$$D=\begin{vmatrix} -z & 1 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & -z & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -z & -2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & +z & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & -a_n & 1 \end{vmatrix}=0.$$

From the Hadamard condition for the non-vanishing of a determinant follows that all the zeros lie in $|z|\leq 3$ if $|a_1|+|a_2|+\cdots+|a_n|<1$.

M. Marden (Milwaukee, Wis.)

2732:

Meĭman, N. N. Zeroes of a certain class of multi-valued functions. Dokl. Akad. Nauk SSSR **124** (1959), 1211–1214. (Russian)

Functions of complex variables of the form $f(z)=g(z)+ih(z)$ are considered, where $g(z)$ and $h(z)$ are single valued and analytic up to a regular point on the boundary of a region G (which contains $z=\infty$ in its interior or on its boundary) and are real on the boundary of G. Further, for each $\eta>0$, $|(g(z)-ih(z))/f(z)|<1+\eta$ for all points z sufficiently far from $z=0$. Let M be the region in G for which $|(g(z)-ih(z))/f(z)|>1$. Then the λ-index of $f(z)$ is defined to be the sum of the multiplicities that the roots of $h(z)/g(z)=\lambda$ have on the boundary of M. The λ-index of $f(z)$ does not exceed the number of zeros of $f(z)$ in G and is equal to the number of zeros if λ is not the asymptotic value of $h(z)/g(z)$ in G.

R. G. Langebartel (Urbana, Ill.)

2733:

Balk, M. B. A theorem of Liouville type. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 65–71. (Russian)

The author's main result is the following theorem: Let $P(x, y)\not\equiv 0$ be a real polynomial of degree s and let D^+, D^- be sets of points in the (x, y) plane where $P(x, y)>0$ [<0, resp.]; if $F(z)$ is a one-valued analytic function in the $z=x+iy$ plane such that Im $F(z)$ is bounded from above [below] in D^+ and bounded from below [above] in D^-, then every isolated singularity of $F(z)$ is a pole of order $\leq s$ and lies on the curve $P(x, y)=0$. In addition the author proves that a pole of multiplicity >1 can lie only in a singular point of $P(x, y)=0$. In the proof of the main theorem an estimate of generalized Laurent series ($\sum_{n=-\infty}^{\infty} p_n(r)e^{in\varphi}$, $p_n(r)$ polynomials in r and r^{-1}) is used.

V. Linis (Ottawa, Ont.)

2734:

Lebedev, N. A. A method for the analytic continuation of power series. Dokl. Akad. Nauk SSSR **125** (1959), 730–732. (Russian)

Let $w(z)=\sum_{n=-1}^{\infty}\alpha_n(z-a)^n$ be a function which maps a simply connected domain G on $|w|>\rho>0$, $a\in G$, $a\neq\infty$; let L_R be the image of $|w|=R>\rho$ under the inverse mapping and let G_R be a domain having L_R as a boundary and containing point a; let (*) $f(z)=\sum_{n=0}^{\infty}a_{n+1}(z-a)^n$ converge in a neighborhood of a; put $w^k=\sum_{n=-k}^{\infty}\beta_n^{(k)}(z-a)^n$, $k=0, 1, 2, \cdots$, and $\gamma_k=\sum_{n=0}^{k}\beta_{-n}^{(k)}a_n$. The author proves that $l=\limsup_{k\to\infty}|\gamma_k|^{1/k}\leq\rho$ [resp. $\leq R$] is a necessary and sufficient condition for the convergence of (*) in G [resp. G_R] to a function $f(z)$ regular there, and having a zero of order ≥ 2 at ∞ if $\infty\in G$ [resp. G_R]. Also

$$f(z)=w'(z)\sum_{k=0}^{\infty}\gamma_k/w^{k+1}$$

for all $z\in G$ if $R\leq\rho$ [resp. $z\in G_R$ if $R>\rho$]. The author remarks that (i) the condition on the order of zero can be removed, and (ii) the theorem can be extended to multiply connected domains. A particular case of the theorem ($a=\infty$) has been proved by Havin [same Dokl. **118** (1958), 879–881; MR **20** #4637].

V. Linis (Ottawa, Ont.)

2735:

Unkelbach, Helmut. Die Vollständigkeit der funktionentheoretischen Invarianten kongruenter Kreisbogenecken. Monatsh. Math. **63** (1959), 32–38.

For generalized circular polygons lying on Riemann surfaces, the author has derived certain invariants with respect to the congruent vertices [Math. Ann. **130** (1955), 327–336; MR **17**, 956]. In the present paper it is shown that there exists essentially no further invariant of this nature.

Y. Komatu (Tokyo)

2736:

Royden, H. L. Rings of meromorphic functions. Proc. Amer. Math. Soc. **9** (1958), 959–965.

The present paper is concerned with algebraic structures associated with compact Riemann surfaces and finite Riemann surfaces. For compact Riemann surfaces it is the field of meromorphic functions which is the object of study; for finite Riemann surfaces it is the field of quotients of bounded analytic functions. The valuations are determined for each of these fields; it is noted that the case of the compact surface is classical. The result may be stated: to each valuation v, there corresponds a unique point p of the surface such that $v(f)=\partial_f(p)$, where $\partial_f(p)$ is the exponent of the divisor of f associated with p. In addition, homomorphisms of such fields (and related ones) are studied and the following results are established. (1) If F_0 is the field of meromorphic functions on a compact Riemann surface W_0 and φ is an algebraic (i.e., constant-preserving) homomorphism of F_0 into the field F_2 of meromorphic functions on a Riemann surface W_2, then there is an analytic map ψ of W_2 into W_0 such that $\varphi(f)=f\circ\psi$. (2) Let W_0 be a compact Riemann surface and W_1 a connected open subset of W_0 with the property that for each boundary point of W_1 there is a bounded analytic function on W_1 having an essential singularity at that point. Let F_1 be the field of quotients of the bounded analytic functions on W_1 and let F_2 be the field of all meromorphic functions on some Riemann surface W_2. Let φ be a ring homomorphism of F_1 into F_2 such that $\varphi(i)=i$ (and $\varphi(f)$ is not constant for some $f\in F_1$). Then there is an analytic map ψ of W_2 into W_1 such that $\varphi(f)=f\circ\psi$.

The author wishes to point out the following corrections. (1) The following hypothesis is to be added to Proposition 4: "For each discrete set $\{p_n\}$ of points on W_1 there exists $f\in F_1$ such that $(f(p_n))$ is not convergent." This hypothesis is used to deduce the continuity of ψ when W_1 is not compact. It is not known whether Proposition 4 remains true without this hypothesis. (2) In Proposition 3, "closure of the" is to be inserted before the next to the

last line. (3) In the next to the last line of Lemma 3 "analytic" is to be replaced by "meromorphic" and "are bounded if h is" of the last line is to be deleted.

M. H. Heins (Urbana, Ill.)

2737:

Peyerimhoff, Alexander; und Richert, Hans-Egon. Über das Anwachsen analytischer Funktionen auf vertikalen Geraden. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 125–134.

The authors generalize a theorem of Lindelöf which expresses the rate of growth in terms of powers of e^t, t, and $\log t$. They replace these functions by functions of "normal oscillation". The methods are different from the usual proof of the Lindelöf theorem.

$\phi(t)$ is called normally oscillating if there exist constants C and γ such that for all y, $t \geq T_0$, $0 < \phi(y) \leq Ce^{\gamma|y-t|}\phi(t)$. It is shown that if $f(s)$, $s=\sigma+it$, is regular in $\sigma_1 \leq \sigma \leq \sigma_2$, $t \geq T_0 \geq 0$, and $f(\sigma+it) = O(e^{Kt})$ there for some K, then the existence of normally oscillating ϕ_1 and ϕ_2 such that

$$|f(\sigma_\nu+it)| \leq \phi_\nu(t), \quad t \geq T_0, \quad \nu = 1, 2,$$

implies

$$f(\sigma+it) = O(\phi_1(t)^{(\sigma_2-\sigma)/(\sigma_2-\sigma_1)} \cdot \phi_2(t)^{(\sigma-\sigma_1)/(\sigma_2-\sigma_1)}).$$

If the above condition on $f(\sigma_\nu+it)$ is replaced by

$$\left(\int_{T_0}^{T} |f(\sigma_\nu+it)|^p dt\right)^{1/p} \leq \phi_\nu(T) \text{ for } T \geq T_0, \quad \nu = 1, 2,$$

for some $p \geq 1$, then

$$\left(\int_{T_0}^{T} |f(\sigma+it)|^p dt\right)^{1/p} = O(\phi_1(T)^{(\sigma_2-\sigma)/(\sigma_2-\sigma_1)} \cdot \phi_2(T)^{(\sigma-\sigma_1)/(\sigma_2-\sigma_1)}).$$

It is also shown that for any $k \geq 0$ and any $\eta(t) > 0$, $\nearrow +\infty$, there is a Laplace integral $\int_0^\infty e^{-su}a(u)du$ with abscissa of (C, k) summability α, such that, for the (C, k) sum $f(s)$,

$$f(\sigma+it) \neq O\left(\frac{t^{k+1}}{\eta(t)}\right)$$

for all $\sigma > \alpha$, which shows that the known $o(t^{k+1})$ cannot be bettered in this sense. A similar result holds for Dirichlet series.

D. Waterman (Milwaukee, Wis.)

2738:

Srivastava, R. S. L. A note on derivatives of integral functions. Ganita **9** (1958), 15–20.

Let $f(z)$ be an entire function of order ρ, lower order λ. Put

$$M(r) = \sup_{|z|=r} |f(z)|, \quad M^{(s)}(r) = \sup_{|z|=r} |f^{(s)}(z)|,$$

$$I = \log \{M^{(s)}(r)/M(r)\}/s \log r.$$

R. P. Srivastav [Math. Student **25** (1957), 11–15; MR **20** #2442; see p. 12] proved $\liminf I = \lambda - 1$ and, if $\lambda > 1$, $\limsup I = \rho - 1$. The author points out that the condition $\lambda > 1$ can be dropped, if r is made to tend to infinity outside a certain set of finite logarithmic length. Several simple corollaries and analogous theorems are proved.

W. H. J. Fuchs (Ithaca, N.Y.)

2739:

Srivastav, R. P. On the fractional derivatives of an integral function. Ganita **9** (1958), 21–25.

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be an entire function. Define, for $\beta > 0$,

$$f^{(\beta)}(z) = \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} a_n z^{n-\beta}.$$

The author extends to these fractional derivatives and their iterations some theorems about the order and the maximum term of ordinary derivatives.

W. H. J. Fuchs (Ithaca, N.Y.)

2740:

Boas, R. P., Jr. A variational method for trigonometric polynomials. Illinois J. Math. **3** (1959), 1–10.

The methods of R. P. Boas and A. C. Schaeffer [Amer. J. Math. **79** (1957), 857–884; MR **20** #972] for investigating certain inequalities satisfied by entire functions of exponential type are developed for trigonometric polynomials. The results are different when the extremal function for the class of entire functions is not a trigonometric polynomial. A comprehensive result on existence and uniqueness of extremal trigonometric polynomials is established. A particular example is developed in detail. Part of the result follows.

If $f(z)$ is a real trigonometric polynomial of order $n > 1$ then the maximum of $|\lambda n^2 f(0) + f''(0)|$ is attained by $\pm \cos nz$ if $\lambda \leq 1/3 + 1/6n^2$ and by a function of the form $\pm \cos\{n \cos^{-1}(\omega \cos z + \omega - 1)\}$ if $\lambda > 1/3 + 1/6n^2$.

Inequalities for ordinary polynomials $p_n(x)$ are derived by considering $p_n(\cos\theta)$ and $p_n(e^{i\theta})$.

A. J. Macintyre (Cincinnati, Ohio)

2741:

Mascart, Henri. Sur la résolution d'une équation fonctionnelle particulière. C. R. Acad. Sci. Paris **248** (1959), 626–629.

Let L be a differential operator applicable to the set of all entire functions of a fixed order and type and such that for integers α and β, $\{L(f)\}^{(\beta)} = L(f^{(\alpha)})$. Let $h(z)$ be an entire function of order σ' and type τ'. For a given pair σ, τ of real numbers, the author investigates entire solutions of the functional equation $L(f) = h(z)$ of order σ and type τ.

R. E. Fullerton (College Park, Md.)

2742:

Tsuji, Masatsugu. On the capacity of a set in the space of regular functions and its applications. Comment. Math. Univ. St. Paul. **7** (1959), 1–12.

Let D denote a Jordan domain in the z-plane, and Ω the family of functions g that are regular in D and satisfy the condition $|g(z)| < 1$. Corresponding to a fixed closed region D' in D, topologize Ω by the metric

$$\rho(g, h) = \max_{z \in D'} |g(z) - h(z)|.$$

Let $\mathfrak{M}$ be a closed set in Ω. For each z in D, each non-negative mass distribution σ that is defined on Borel sets in $\mathfrak{M}$ (with $\sigma(\mathfrak{M}) = 1$) determines a potential function

$$\chi_\sigma(\zeta) = \int_{\mathfrak{M}} \{-\log|\zeta - g(z)|\} d\sigma(g);$$

this in turn leads to the notion of capacity of closed sets and, more generally, of Borel sets in $\mathfrak{M}$.

The author establishes analogues of Picard's theorem. Examples: (1) Let E be a closed set in D', of logarithmic capacity 0. Let $w(z)$ be meromorphic in $D^*=\overline{D'}-E$, and let each point of E be an essential singularity of $w(z)$. Then, for all g in $\mathfrak{R}$ except those of a set of capacity 0, the function $w(z)-g(z)=0$ has infinitely many zeros in D^*. (2) Let w be meromorphic and of unbounded characteristic in the unit disk, and let the domain D above be a disk $|z|<R$ $(R>1)$. Then the function $w(z)-g(z)$ has infinitely many zeros in $|z|<1$, for all g in $\mathfrak{R}$ except a set of capacity 0. A similar result concerns functions of Seidel's class (A). *G. Piranian* (Ann Arbor, Mich.)

2743:

Schubart, Hans. Zur Wertverteilung der elliptischen Funktionen und der Painlevéschen Transzendenten als Lösungen algebraischer Differentialgleichungen. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 161–186.

The author continues his studies on the Painlevé differential equations $w''=6w^2+A_0z+B_0$ and $w''=2w^3+(A_1z+B_1)w+C$ [cf. H. Schubart, Arch. Math. **7** (1956), 284–290; MR **18**, 472; H. Schubart und H. Wittich, Math. Z. **66** (1957), 364–370; MR **19**, 24]. These are the only equations of the form $w''=P(z, w)$, P polynomial in z and w, the solutions of which are single-valued meromorphic functions for $z\neq\infty$.

The author draws conclusions on the value distribution properties of the solutions by directly applying methods of the Nevanlinna theory to the equations. Several results concerning deficient values, ramification, etc. are obtained, e.g., that the solutions possess completely ramified finite values if and only if $A_0=A_1=0$. *O. Lehto* (Helsinki)

2744:

Kulshrestha, P. K. On measure of curvature of level curves and orthogonal trajectories of a class of mean p-valent functions in the unit circle. Ganita **9** (1958), 1–4.

The author develops formulas for the quantities indicated in the title. However, the reviewer is unable to understand the author's equation (4), and the formulas obtained appear to be incorrect because they seem to fail for the mean p-valent function $w=z^p$.

A. W. Goodman (Lexington, Ky.)

2745:

Dinghas, Alexander. Verzerrungsschranken bei schlichten Abbildungen des Einheitskreises. Arch. Math. **8** (1957), 413–416.

Let S be the class of functions $f(z)=z+a_2z^2+\cdots$ univalent in $|z|<1$. Then if Bieberbach's conjecture $|a_k|\leq k$ holds for $k=1$ to n and all functions $f(z)\in S$, then the sharp inequality

$$|f^{(n)}(z)| \leq \frac{n!(n+|z|)}{(1-|z|)^{n+2}}$$

also holds for $f(z)\in S$. This result appears to be due to F. Marty [C. R. Acad. Sci. Paris **194** (1932), 1308–1310] who carried out the detailed proof only in the cases $n=2$, 3. The present author, who seems unaware of Marty's paper, now gives the proof in full generality.

W. K. Hayman (London)

2746:

Sinclair, Annette. A general solution for a class of approximation problems. Pacific J. Math. **8** (1958), 857–866.

An abstract method is used to deal with a class of problems on approximation and thus to obtain some new results. The decomposition $W_1\cup W_2\cup\cdots$ of a set S is introduced by means of its relation to an increasing sequence $\{R_i\}$ $(i=1, 2, \cdots)$ of R-covering sets, where R is an open subset of a topological space, $R=\bigcup_{i=1}^{\infty}R_i$, $\bar{R}_i$ interior to R_{i+1}, $W_i\subset R_i$, but $W_{i+1}\cap R_i$ is empty, therefore $W_i\cap W_k$ $(i\neq k)$ empty also. Classes of functions $\mathfrak{B}_n$ and $\mathfrak{R}_n$ are considered which transform W_n and R_n, respectively, into domains of the complex plane. It is shown (Theorem 1) that, given a function $w(X)$ defined on S and $\varepsilon_i>0$, there exists a function $r(X)$ on R such that $|r(X)-w(X)|<\varepsilon_i$ when $X\in W_i$, $i=1, 2, \cdots$, assuming certain properties (i) of convergence for sequences of functions on R_{n+1} and (ii) concerning approximation for functions defined on R_n and W_{n+1} simultaneously. Clearly the result is sharper than a statement

$$|r(X)-w(X)| < \varepsilon$$

on S, where $\varepsilon>0$ is given.

The theorem is used to deduce a result on Q-sets [cf. Sinclair, Trans. Amer. Math. Soc. **72** (1952), 148–164; MR **13**, 832], a theorem on approximation on a set S consisting of an infinity of circular discs touching the real axis; and finally an extension of Carleman's approximation theorem, with the values of the approximating functions at certain points being preassigned.

In Part II a topological abstraction is made of theorem 1, so that it can be considered as a special case of a general result (theorem 5). *H. Kober* (Birmingham)

2747:

Chen, Kien-kwong. Uniform approximation by integral functions of the order ρ to the functions on a Jordan region of the index ρ. Sci. Sinica **6** (1957), 1–20.

Results stated in a previous paper [Sci. Record (N.S.) **1** (1957), no. 1, 19–23; MR **20** #3408] are proved. The problem is the uniform approximation by integral functions of finite types of the order ρ to a given function $f(z)$ which need not be bounded, both on the real axis (see theorems 1 and 2) and on certain Jordan regions K_σ, among them the strip (theorems 3–8), where $f(z)$ is required to satisfy a condition of continuity. Theorem 1, for instance, reads:

Let $f(x)$ be integrable on any finite interval, $=O(|x|^p)$ $(|x|\to\infty$, $p=1$, or 2, 3, $\cdots)$, let $\omega(s)$ $(s>0)$ be positive, monotone and $\omega(+0)=0$, $s\omega'(s)/\omega(s)\leq A$, $\omega(s)=O(s^p)$ $(s\to\infty)$, and let

$$m_k(x, x+h) = h^{-k}\Gamma(k)^{-1}\int_0^h (h-t)^{k-1}|f(x+t)-f(x)|dt$$

be $\leq\omega(h)$ $(-\infty<x<\infty)$; then there are integral functions $g_\sigma(x)$ of exponential type σ $(\sigma\to\infty)$ such that, in $(-\infty, \infty)$, $|f(x)-g_\sigma(x)|\leq C\omega(\sigma^{-1})$, where C depends on f only.

The approximating functions g_σ are explicitly given in terms of f and the Mittag-Leffler functions. {The author has, however, overlooked the fact that there exists a theory of approximation to non-bounded functions of a

wide class [H. Kober, Trans. Amer. Math. Soc. **61** (1947), 293–306; MR **8**, 455] apparently not covered by his, otherwise rather general, results.}

H. Kober (Birmingham)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 2836.

2748:

Siegel, C. L. On meromorphic functions of several variables. Bull. Calcutta Math. Soc. **50** (1958), 165–168.

Soit C une variété analytique compacte à n dimensions complexes, K le corps des fonctions méromorphes sur C. On établit: (1) $n+1$ fonctions $f, f_1, \cdots, f_n$ de K sont algébriquement dépendantes sur le corps des nombres complexes—(2) s'il existe n fonctions $f_1, \cdots, f_n$ algébriquement indépendantes sur K, K est une extension algébrique du corps des fonctions rationnelles de $f_1, \cdots, f_n$. La démonstration est remarquablement élémentaire. On considère un nombre fini de points $a_1, \cdots, a_N$ sur C et on construit un polynome $P(x, x_1, \cdots, x_n) \not\equiv 0$ tel que $P(f, f_1, \cdots, f_n)$ ait ses dérivées nulles en a_k jusqu'à l'ordre n_k; il en résulte d'après le lemme de Schwarz une majoration de $|P|$ sur C; une estimation du nombre de conditions linéaires imposées montre la possibilité de choisir $P \not\equiv 0$, la majoration obtenue pour $|P(f, f_1, \cdots, f_n)|$ entraînant que cette quantité soit identiquement nulle sur C.

P. Lelong (Paris)

2749:

Koecher, Max. Die Geodätischen von Positivitätsbereichen. Math. Ann. **135** (1958), 192–202.

The author continues his study of domains of positivity [started in Amer. J. Math. **79** (1957), 575–596; MR **19**, 867]. He considers a self-dual open convex cone $P \subset R^n$. This generalizes the symplectic geometry of Cartan and Siegel, where P is the set of positive definite real bilinear forms. There is a natural Riemannian metric, which is invariant under the linear automorphisms of P. In the present paper properties of the geodesic of this metric are shown to lead to important algebraic facts about P. There is a commutative bilinear operation O on R^n which arises from the geometry of P. Under a homogeneity condition (satisfied if the linear automorphisms of P are transitive) R^n becomes a Jordan algebra under O. Then the unit e is an element of P which is fixed under the previously defined involution $x \to x^*$ on P generalizing the map $y \to y^{-1}$ in the symplectic case. The geodesics through e are given by $g(t) = \exp(tx) = e + \sum_1^\infty (n!)^{-1}(tx)^n$ for $x \in R^n$, and exp is a 1-1 map of R^n onto P. Thus the geodesic joining two points is unique. Also $x^* = x^{-1}$ for $x \in P$, and x^* is rational in x. Continuing the analysis of the norm N and gamma factor Γ started in the first paper, the author now shows that N is the square root of a polynomial. Hence an argument of Bochner shows that Γ is an exponential multiplied by ordinary Γ functions. Some of the results have been found independently by O. Rothaus [Bull. Amer. Math. Soc. **64** (1958), 85–86; MR **20**#3159].

The constructions of this paper form an important natural link between a certain subclass (not all!) of the bounded symmetric complex domains of Cartan and certain Jordan algebras of various bilinear forms. This helps explain the relevance of Cartan's spaces to the analytic theory of bilinear forms.

D. Lowdenslager (Princeton, N.J.)

2750:

★**Séminaire Henri Cartan; 10e année: 1957/1958. Fonctions Automorphes.** 2 vols. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. ii+214+ii+152 pp. (mimeographed)

This tenth year of the Cartan seminar notes contains, as one has come to expect with this series, the exposition of a considerable amount of recent and interesting mathematics. The unifying theme for the 1957/58 year of the seminar was automorphic functions on the Siegel pattern; the five variations on this theme corresponding more or less to the various lecturers participating are, briefly and in the order of their appearance, as follows.

(A) Discrete subgroups of some classical groups and their fundamental domains (2 lectures by A. Weil).

(B) A review of some of Siegel's work as background for the remainder of the seminar (2 lectures by H. Cartan).

(C) The Siegel modular forms examined and generalized from the point of view of harmonic analysis and group representations (6 lectures by R. Godement).

(D) The compactification of the Siegel quotient spaces (7 lectures by H. Cartan and I. Satake).

(E) A connection between modular functions and the moduli of abelian varieties (3 lectures by G. Shimura).

These sections will next be reviewed separately in somewhat more detail.

(A) Weil's lectures. The first lecture is a review of work of Minkowski and Siegel on the reduction of quadratic forms under arithmetic equivalence [cf. C. L. Siegel, Abh. Math. Sem. Hamburg Univ. **13** (1940), 209–239; MR **2**, 148]. This can be considered as a case of the following general problem: to find a decent fundamental domain for a discrete subgroup Γ of a semi-simple Lie group G, in terms either of the group G itself or of the symmetric space G/K, where K is a maximal compact subgroup of G. In this case G is the projective linear group (the quotient of $\mathrm{GL}(n, R)$ by its center), Γ is the image in G of the subgroup of integer matrices in $\mathrm{GL}(n, R)$, and K is the image in G of the orthogonal subgroup $\mathrm{O}(n, R) \subset \mathrm{GL}(n, R)$; the homogeneous space $\mathrm{GL}(n, R)/\mathrm{O}(n, R)$ can be identified with the cone P of positive definite symmetric matrices A (or positive definite quadratic forms) under the group action $A \to {}^tXAX$ for $X \in \mathrm{GL}(n, R)$, and G/K is the image of the cone P in projective space. The work of Minkowski furnishes an explicit fundamental domain for the action of Γ on G/K. The work of Siegel demonstrates that Γ is what Weil terms Minkowskian in G: that is, there is an open subset $U \subset G$ of finite measure such that $KU = U$, that $U\Gamma = G$, and that for any $x \in G$ for which $x\Gamma x^{-1} \cap \Gamma$ has finite index both in $x\Gamma x^{-1}$ and in Γ there are only finitely many $\gamma \in \Gamma$ for which $Ux\gamma \cap U \neq \emptyset$.

In the second lecture two further cases of the above general problem are discussed. The first is that in which G is the orthogonal group of a non-singular indefinite quadratic form F, and Γ is the group of integer matrices in G when G is expressed in a matrix form; the second is the corresponding problem for a non-singular skew-symmetric form F. In both it can be shown that Γ is

Minkowskian, by identifying the homogeneous space G/K with a submanifold of the cone P of positive definite symmetric matrices and using the results discussed in the first lecture. The proof is carried out in some detail for the first case; the connection between the second case and the Siegel spaces is also discussed in some detail. {The general problem has subsequently been treated more fully in mimeographed lecture notes by Weil [*Discontinuous subgroups of classical groups*, Univ. of Chicago, 1958].}

(B) Cartan's lectures. These two lectures form a review of background material required for the remaining lectures, taken principally from the two fundamental papers of Siegel on automorphic functions [Math. Ann. **116** (1939), 617–657; Amer. J. Math. **65** (1943), 1–86; MR **1**, 203; **4**, 242]; for the present purpose these two lectures need only be discussed to establish notation. The Siegel space S_n is the space of symmetric complex matrices z with positive definite imaginary parts ($\mathrm{Im}\ z \gg 0$). This is a homogeneous space under the symplectic group $\mathrm{Sp}(n, R)$, the group of real $2n \times 2n$ matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that ${}^tMJM = J$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; the group action is defined by $M: z \to (az+b)(cz+d)^{-1}$. The most important discontinuous subgroup of $\mathrm{Sp}(n, R)$ for the present purpose is the symplectic modular group $\Gamma = \Gamma_n$ consisting of the integral symplectic matrices; another discontinuous subgroup of some interest is the paramodular group Γ_δ associated to a diagonal integral matrix δ, consisting of those symplectic matrices for which a, $b\delta^{-1}$, δc, $\delta d\delta^{-1}$ are integral. Siegel's explicit form for the fundamental domain of Γ in S_n is described, thus completing the previous lecture by showing that Γ and Γ_δ are Minkowskian in $\mathrm{Sp}(n, R)$. A factor of automorphy $R(M, z)$ for Γ is a mapping from $\Gamma \times S_n$ into the complex Lie group of automorphisms of a complex vector space F, which is analytic in z and which satisfies the functional equation

$$R(M_1M_2, z) = R(M_1, M_2z)R(M_2, z);$$

to any complex analytic representation ρ of $\mathrm{GL}(n, C)$ in the vector space F is associated a factor of automorphy $J_\rho(M, z) = \rho(cz+d)$. A modular form (or automorphic form) for this factor of automorphy is an analytic mapping $f(z)$ of S_n into F such that $f(Mz) = J_\rho(M, z)f(z)$ for all $M \in \Gamma$. An analysis of the Fourier expansion of a modular form shows that there is a canonical linear transformation $\Phi_{n,r}$ from the space of modular forms for Γ_n on S_n into the space of modular forms for Γ_r on S_r, $r \leq n$.

(C) Godement's lectures. Lecture 5 and the initial part of lecture 6 are devoted to two essential preliminaries. First is a brief yet lucid discussion of the irreducible representations ρ of the group $\mathrm{GL}_+(n, R)$ (real $n \times n$ matrices of positive determinant) in a finite-dimensional complex vector space F_ρ; the concept of the highest weight of the representation is the central theme of this discussion, from which only the notation α_n, for the power to which the determinant occurs in the highest weight, will be needed in this review. If α_n is real, as will be assumed henceforth, there is a positive definite Hermitian inner product $\langle a, b \rangle$ on F_ρ such that $\rho({}^tg) = \rho(g)^*$ for all $g \in \mathrm{GL}_+(n, R)$, where $\rho(g)^*$ is the adjoint of $\rho(g) \in F_\rho$ in this Hermitian structure; the norm on F_ρ associated to this inner product is $\|a\| = \langle a, a \rangle^{1/2}$. If α_n is an integer the representation ρ extends to a complex analytic representation of $\mathrm{GL}(n, C)$, to which is associated a factor of automorphy $J_\rho(M, z)$ as above. Second is the discussion of convergence conditions for certain integrals which will arise later; these are of the form

$$(1) \qquad \int_Y \|\rho(y^{1/2})a\|^\sigma \exp(-\pi \,\mathrm{Tr}(sy))\,(\det y)^{-\alpha-1}\,dy,$$

where Y is the space of real symmetric positive definite matrices y, for which of course $y^{1/2}$ is well-defined, $a \in F_\rho$ is arbitrary, $\sigma \geq 1$, s is a real symmetric matrix, and dy is the usual Euclidean measure. For one-dimensional representations these integrals have been investigated by Siegel [Ann. of Math. (2) **36** (1935), 527–606]. It is proved that the integral (1) converges if and only if s is positive definite and $\alpha_n > 2n/\sigma$; the second condition is needed only to ensure the convergence of that part of the integral extended over the set of matrices y with $\det y \leq c$ for any $c > 0$. In the proof Y is considered as a homogeneous space of the group $\mathrm{GL}_+(n, R)$ and the integral is rewritten as an integral over the group, in the study of which the work lies; an appendix to lecture 5 contains a general discussion of integrals and invariant measures on groups and homogeneous spaces, including especially such relations as arise in this proof and again later in lecture 10. As a first application of these convergence conditions it is noted that the integral

$$H_\rho(s) = \int_Y \rho(y) \exp(-\pi \,\mathrm{Tr}(sy))\,(\det y)^{-\alpha-1}dy$$

converges whenever s is positive definite and $\alpha_n > n$, and is then a positive definite Hermitian operator on F_ρ; the evaluation of this integral is simplified by observing that

$$H_\rho(s) = (\det s)^{(\alpha+1)/2}\rho(s^{1/2})H_\rho(1)\rho(s^{1/2}),$$

but an explicit value for $H_\rho(1)$ is given only for one-dimensional representations ρ, in which case it was determined by Siegel [loc. cit.].

The remainder of lecture 6 is devoted to a study of the space $H^2(\rho)$ of holomorphic functions $f(z)$ on the Siegel space S_n with values in F_ρ for which

$$(2) \qquad \|f\|_2 = \left(\int_{S_n} \|\rho(y^{1/2})f(z)\|^2 dz\right)^{1/2} < +\infty,$$

where dz is the group-invariant measure on S_n and $y = \mathrm{Im}(z)$; this is a Hilbert space with the inner product corresponding to (2). The two central themes of this study are the Fourier transform and the reproducing kernel. The set of the real parts x of matrices $z \in S_n$ is the entire Euclidean space of real symmetric matrices; by using the analyticity of a function $f(z) \in H^2(\rho)$ its Fourier transform with respect to x can eventually be written in the form

$$(3) \qquad f(s) = \int f(z) \exp(-2\pi i \,\mathrm{Tr}(sz))dx,$$

where s is a real symmetric matrix and the integral, not being absolutely convergent, is interpreted in an L^2-sense. The Plancherel formula and the convergence conditions obtained earlier for the integral (1) show that $H^2(\rho) \neq 0$ if and only if $\alpha_n > n$, and that $f(s) \neq 0$ only when $s \gg 0$; indeed the Fourier transform (3) is an isomorphism of

$H^2(\rho)$ onto the Hilbert space of those measurable functions $f(s)$ of positive definite symmetric matrices s for which

$$\int_{s>0} \|H_\rho(4s)^{1/2} f(s)\|^2 ds < +\infty,$$

and in the inverse isomorphism

$$(4) \qquad f(s) \to f(z) = \int_{s>0} f(s) \exp(2\pi i \operatorname{Tr}(sz)) ds$$

the integral is absolutely convergent for all $z \in S_n$. If the Hilbert space $H^2(\rho)$ is non-empty ($\alpha_n > n$) it must have a Bergmann kernel function $K_\rho(z_1, z_2)$, that is, a linear operator on F_ρ holomorphic in z_1 and $\bar{z}_2$ with the property that

$$(5) \qquad f(z_1) = \int_{S_n} K_\rho(z_1, z_2) \rho(y_2) f(z_2) dz_2$$

for all $f(z) \in H^2(\rho)$, the integral being absolutely convergent. By considering the unitary representation $L_\rho(M^{-1})$: $f(z) \to J_\rho(M, z)^{-1} f(Mz)$ of $\mathrm{Sp}(n, R)$ on $H^2(\rho)$ a functional equation is derived for the kernel function which shows that it must indeed have the form

$$(6) \qquad K_\rho(z_1, z_2) = c(\rho) \rho\left(\frac{z_1 - \bar{z}_2}{2i}\right)^{-1}$$

for some constant $c(\rho)$; the introduction of the factor of automorphy $J_\rho(M, z)$ of course limits the discussion to representations ρ with integral α_n, although the modifications necessary to cover the case of arbitrary real α_n are briefly sketched.

Lecture 7 begins the study of generalized modular forms proper; it and the two following lectures are quite independent of all of lectures 5 and 6 except the initial discussion of the representation ρ. For the definition of the generalized modular forms consider in addition to the representation ρ (with α_n integral) a linear representation μ of the symplectic modular group Γ in a finite-dimensional complex vector space F_μ; it will be assumed throughout that the kernel of μ is of finite index in Γ, so that a Hermitian inner product can be put on F_μ in terms of which μ is a unitary representation. A modular form (or automorphic form) of type (ρ, μ) is then a holomorphic function $f(z)$ on the Siegel space S_n with values in the complex vector space $F = \mathrm{Hom}(F_\mu, F_\rho)$ such that

$$(7) \qquad f(Mz) = J_\rho(M, z) f(z) \mu(M^{-1})$$

for all $M \in \Gamma$. Many of the results in these lectures are derived for more general discontinuous subgroups of $\mathrm{Sp}(n, R)$ than Γ, but for brevity such extensions will not be noted in this review. As in the case of the classical modular forms the first step is to obtain a Fourier series expansion for these generalized modular forms, which is of the type

$$(8) \qquad f(z) = \sum_s f(s) \exp(2\pi i \operatorname{Tr}(sz)),$$

where the summation is extended over some set of rational symmetric matrices s; in particular if μ is trivial only semi-integral matrices s appear. The functional equation (7) applied to this Fourier expansion leads to the Koecher theorem that $f(s) = 0$ unless s is positive semi-definite; in addition the Fourier coefficients $f(s)$ for positive semi-definite matrices of rank $< n$ are subject to some further restrictions, the investigation of which (at the end of lecture 7 and the beginning of lecture 8) is necessary in order to extend the operators $\Phi_{n,r}$ to these generalized modular forms. A modular form is called a cusp form (Spitzenform) if $f(s) = 0$ unless s is positive definite.

The fundamental tool for the remainder of lectures 7, 8, and 9 is the inner product for modular forms of type (ρ, μ) generalizing that introduced and exploited so successfully in one complex variable by Petersson [probably most readily accessible in: Jber. Deutsch. Math. Verein. **49** (1939), 49–75]. The space $F = \mathrm{Hom}(F_\mu, F_\rho)$ has a natural Hermitian inner product $\langle a, b \rangle$ induced by the inner products on F_μ and F_ρ, and an associated norm $\|a\| = \langle a, a \rangle^{1/2}$; the inner product of two generalized modular forms $f(z)$ and $g(z)$ is then defined by

$$(9) \qquad \langle f, g \rangle_\Gamma = \int_{S_n/\Gamma} \langle \rho(y^{1/2}) f(z), \rho(y^{1/2}) g(z) \rangle dz$$

whenever the integral converges, where dz is the group-invariant measure on S_n as before. To simplify questions of the convergence of this inner product introduce the spaces $H^p_\Gamma(\rho, \mu)$ of those modular forms $f(z)$ for which

$$\int_{S_n/\Gamma} \|\rho(y^{1/2}) f(z)\|^p dz < +\infty \quad (p = 1, 2, \cdots),$$

and the space $H^\infty_\Gamma(\rho, \mu)$ of those modular forms for which $\|\rho(y^{1/2}) f(z)\|$ is bounded on S_n. These are Banach spaces with the obvious norms, $H^\infty_\Gamma(\rho, \mu) \subset H^p_\Gamma(\rho, \mu) \subset H^1_\Gamma(\rho, \mu)$, and the inner product (9) is of course a (finite) duality between $H^p_\Gamma(\rho, \mu)$ and $H^q_\Gamma(\rho, \mu)$ whenever $1/p + 1/q = 1$; in particular $H^2_\Gamma(\rho, \mu)$ is a Hilbert space with the inner product (9). As for the significance of these spaces in the study of modular forms, note that the inner product (9) is always convergent if one of the two forms is a cusp form; indeed the cusp forms are contained in all of the spaces $H^p_\Gamma(\rho, \mu)$ ($1 \leqq p \leqq \infty$). If $\alpha_n \neq 0$ the space $H^\infty_\Gamma(\rho, \mu)$ is precisely the space of cusp forms; in an appendix to lecture 9 Satake proves, using results to be discussed in part (D), that $H^p_\Gamma(\rho, \mu)$ is also precisely the space of cusp forms if $p\alpha_n > 2n$. The first application of this inner product is in the Hilbert space structure of $H^2_\Gamma(\rho, \mu)$. This Hilbert space, or indeed any closed subspace $H \subset H^2_\Gamma(\rho, \mu)$, must possess a Bergmann kernel function $K_H(z_1, z_2)$ just as the space $H^2(\rho)$ considered earlier; if the subspace H lies in $H^\infty_\Gamma(\rho, \mu)$ as well, then its kernel function $K_H(z_1, z_2)$ is of Hilbert-Schmidt type, which leads to the conclusion that H is finite-dimensional. In particular therefore the space of cusp forms is finite-dimensional; and an inductive use of the operator $\Phi_{n,n-1}$ shows that the space of all modular forms is itself finite-dimensional. Two most interesting digressions, or rather amplifications, are inserted at this point in lecture 8. The first presents the last argument above in a more general setting; the second recalls some of the fascinating questions arising in a study of functions similar to modular forms but which, rather than being complex analytic, are other eigenfunctions of invariant differential operators on Lie groups. This range of problems will doubtless soon be the focus of even more interest, following the suggestive work of Maass [Math. Ann. **121** (1949), 141–183; J. Indian Math. Soc. (N.S.) **20** (1956), 117–162; MR **11**, 163; **19**, 252], Harish-Chandra [Trans. Amer. Math. Soc. **70** (1951), 28–96; MR **13**, 428; etc.], and Selberg [J. Indian Math. Soc. (N.S.) **20** (1956), 47–88; MR **19**, 531]. To return to course again, the second application of the inner product

is in the study of Eisenstein series, which is the topic of lecture 9; the method is a direct extension of Petersson's technique [loc. cit.]. It should perhaps be noted that Petersson and most others have called such series Poincaré series. Consider a positive definite real symmetric matrix s and an element $\omega \in F$ such that

$$\omega\mu(n) = \exp(-2\pi i \operatorname{Tr}(sn))\omega$$

for all translations $z \to z+n$ in Γ, that is, for all symmetric integral matrices n; the Eisenstein series to be considered are of the form

$$(10)\quad E_{s;\omega}(z) = \sum_{M} \exp(2\pi i \operatorname{Tr}(s \cdot Mz))J_\rho(M, z)^{-1}\omega\mu(M),$$

where the summation is extended over a set $\{M\}$ of left coset representatives of Γ modulo the subgroup of translations. If μ is trivial then $E_{s;\omega}(z) = E_s(z)\omega$ where

$$(11)\quad E_s(z) = \sum_{M} \exp(2\pi i \operatorname{Tr}(s \cdot Mz))J_\rho(M, z)^{-1}.$$

The series (10) converges to a cusp form of type (ρ, μ) whenever $\alpha_n > 2n$; and if $f(z)$ is any other cusp form of the same type with the Fourier expansion (8) then $\langle f, E_{s;\omega}\rangle = \operatorname{Tr}(\omega^* H_\rho(4s)\hat{f}(s))$, which leads to the conclusion that every cusp form can be expressed as a linear combination of these Eisenstein series. Godement also derives formulas for the Fourier coefficients of the Eisenstein series expressed in terms of some generalized Bessel functions

$$J_\rho(s) = \int \exp(-2\pi i \operatorname{Tr}(sx + sz^{-1}))\rho(z)^{-1}dx;$$

these reduce to Petersson's formulas for the Fourier coefficients in terms of the classical Bessel functions in one complex variable [Acta Math. 58 (1932), 169–215]. Series similar to (10) can be associated to matrices s which are only positive semi-definite; convergence conditions for such series are investigated, the situation being rather more complicated than in the positive definite case.

The first half of lecture 10 is a continuation of the investigations in lectures 5 and 6 of spaces of holomorphic integrable functions on the Siegel space S_n; the earlier results are extended by considering two other representations for these function spaces. First to any function $f(z)$ of $z \in S_n$ associate the function $f(M)$ of $M \in G = \operatorname{Sp}(n, R)$ defined by

$$(12)\quad f(M) = J_\rho(M^{-1}, i)^{-1}f(M^{-1}i),$$

where i stands for the matrix $iI \in S_n$ which is fixed under the subgroup $K \subset G$, $K \cong U(n)$; this establishes a one-to-one correspondence between the set of all functions on S_n and the set of those functions on the group G for which

$$(13)\quad f(WM) = \rho(W)f(M)$$

whenever $W \in K$ and $M \in G$. In this representation of the function space the norm (2) becomes

$$\|f\|_2 = \left\{\int_G \|f(M)\|^2 dM\right\}^{1/2},$$

where dM is the group-invariant measure on G; and if $K_\rho(M)$ is the function on G associated to the Bergmann kernel function $K_\rho(z, i)$ of $H^2(\rho)$ then (5) becomes

$$(14)\quad f(M_1) = \int_G K_\rho(M_1M_2^{-1})f(M_2)dM_2 = (K_\rho * f)(M_1),$$

with the usual notation for the convolution on G. One can of course also introduce the Banach spaces $L^p(\rho)$ of measurable functions on G satisfying (13) and

$$\|f\|_p = \left\{\int_G \|f(M)\|^p dM\right\}^{1/p} < +\infty;$$

and the Banach subspaces $H^p(\rho) \subset L^p(\rho)$ consisting of those functions on G which are associated to holomorphic functions on S_n through the correspondence (12). A further digression at this point is devoted to Harish-Chandra's direct characterization of the subspaces $H^p(\rho)$ as the spaces of solutions of certain invariant differential equations on the group. The second representation for these function spaces simply involves the bounded homogeneous domain equivalent to the Siegel space; as is well known this domain is the generalized unit disc consisting of the complex symmetric matrices ζ for which $I - \zeta\bar{\zeta}$ is positive definite. In this representation it is proved that for $\alpha_n > 2n$ every function $f(\zeta)$ holomorphic and bounded in the generalized unit disc lies in $H^p(\rho)$ for all p; in particular the polynomials in ζ form an everywhere dense linear subspace of $H^p(\rho)$. These two representations are now used to extend the main results of lectures 5 and 6, those involving the reproducing kernel and the Fourier transform, from the space $H^2(\rho)$ to the space $H^1(\rho)$. Thus it is proved that if $\alpha_n > 2n$ then (14) holds for every $f(M) \in H^1(\rho)$, which in turn leads to the conclusion that $H^1(\rho) \subset H^2(\rho)$; therefore the Fourier transform defined in $H^2(\rho)$ can be applied to any function in $H^1(\rho)$. Moreover the Fourier transform of a function $f(z) \in H^1(\rho)$ is equal almost everywhere to a continuous function $\hat{f}(s)$ which vanishes outside the set of positive definite matrices and which satisfies an inequality of the form

$$\|\hat{f}(s)\| \leq \gamma(\rho)\|f\|_1(\det s)^{-(n+1)/2}\|\rho(s^{1/2})\|,$$

for a constant $\gamma(\rho)$ independent of $f(z)$; this inequality can be used to justify an application of the Poisson summation formula, leading to the result that for $\alpha_n > 2n$ and $f(z) \in H^1(\rho)$

$$(15)\quad \sum_{\substack{n=\text{integral}\\ (n \text{ integral})}} f(z+n) = \sum_{\substack{s=\text{half-integral}\geq 0\\ (s \text{ half-integral})}} \hat{f}(s)\exp(2\pi i \operatorname{Tr}(sz)),$$

both sides being continuously convergent on the Siegel space S_n.

All of the results discussed in the last paragraph are valid for analytic functions with values in $F = \operatorname{Hom}(F_\mu, F_\rho)$ just as for the analytic functions with values in F_ρ, indeed with the same notation. Now considering the functions on G induced by (12), the generalized modular forms of type (ρ, μ) are characterized as the invariants of the linear representation $L_{\rho,\mu}$ of Γ defined by

$$L_{\rho,\mu}(M_0): f(M) \to f(MM_0)\mu(M_0^{-1}), \quad (M \in G, M_0 \in \Gamma),$$

while the spaces $H^p{}_\Gamma(\rho, \mu)$ discussed in lectures 7, 8, 9 are characterized as the spaces of those modular forms for which $\int_{G/\Gamma} \|f(M)\|^p dM < -\infty$. After these observations the second half of lecture 10 contains a discussion of the application of the results obtained in the first half of the lecture to modular forms, thus bringing together the two separate branches of Godement's lectures. The first application is to the representation of modular forms by Poincaré series. For every function $f(M) \in H^1(\rho)$ the series $f^\Gamma(M) = \sum_{M_0 \in \Gamma} f(MM_0^{-1})\mu(M_0)$ converges continuously and represents a modular form $f^\Gamma(M) \in H^1{}_\Gamma(\rho, \mu)$.

The resulting linear map from $H^1(\rho)$ to $H^1{}_\Gamma(\rho, \mu)$ is continuous, and since $H^1{}_\Gamma(\rho, \mu)$ is finite-dimensional the image is the same as the image of any everywhere dense linear subspace of $H^1(\rho)$; for the particular subspace consisting of the polynomials in the representation of S_n as a bounded homogeneous domain, these series are called Poincaré series. The final result is that for $\alpha_n > 2n$ every cusp form (or equivalently every modular form in $H^1{}_\Gamma(\rho, \mu)$) is a Poincaré series and conversely. The second application is to an explicit representation of the Bergmann kernel function $K_\rho{}^\Gamma(M_1, M_2)$ of the subspace

$$H^\infty{}_\Gamma(\rho, \mu) \subset H^2{}_\Gamma(\rho, \mu);$$

to simplify these considerations Godement henceforth assumes that μ is trivial. If $\alpha_n > 2n$ the kernel function $K_\rho{}^\Gamma(M_1, M_2)$ can be expressed in terms of the kernel function $K_\rho(M)$ of $H^2(\rho)$ by

$$K_\rho{}^\Gamma(M_1, M_2) = \sum_{M \in \Gamma} K_\rho(M_1 M M_2{}^{-1}). \tag{16}$$

The integral operator defined by this kernel function is really an orthogonal projection onto $H^\infty{}_\Gamma(\rho, \mu)$, hence the dimension of $H^\infty{}_\Gamma(\rho, \mu)$ is the trace of this operator; and it eventually follows as a corollary of (16) that

$$\dim_C H^\infty{}_\Gamma(\rho, \mu) = \int_{G/\Gamma} \sum_{M \in \Gamma} \operatorname{Tr}(K_\rho(M M_0 M^{-1}))dM.$$

This is a version of the Selberg trace-formula [Selberg, loc. cit.]. The third application is to the relationship between the two explicit representations of modular forms which have been considered, that by Eisenstein series and that by Poincaré series. By use of (15) it is proved that if $\alpha_n > 2n$, $f(z) \in H^1(\rho)$, μ is again trivial, and if in addition

$$\sum_{\substack{s > 0 \\ (s \text{ half-integral})}} \|\rho(s^{1/2}) \hat{f}(s)\| \, (\det s)^{(n+1)/2} < +\infty,$$

then

$$f^\Gamma(z) = \sum_{\substack{s > 0 \\ (s \text{ half-integral})}} E_s(z) \hat{f}(s),$$

where $E_s(z)$ was defined in (11), the series being continuously convergent for $z \in S_n$. Moreover by using the explicit formulas (6) and (16) for the kernel function it follows that

$$K_\rho{}^\Gamma(z_1, z_2) = \sum_{\substack{s > 0 \\ (s \text{ half-integral})}} E_s(z_1) H_\rho(4s)^{-1} \exp(-2\pi i \operatorname{Tr}(s \bar{z}_2)).$$

(D) Cartan's and Satake's lectures. After a brief survey of the necessary background in complex analytic spaces (cf. Séminaire H. Cartan 1953/54 [MR **19**, 577], where such spaces are called "espaces analytiques généraux"), lecture 11 (by Cartan) is devoted to the following problem concerning the analytic continuation of these spaces: if X is a locally compact space and V is an everywhere-dense open subset which has the structure of a normal complex analytic space, when is X itself a normal complex analytic space, compatible with V? The last restriction means that the local ring A_x of holomorphic functions at any point $x \in X$ must be the local ring of those continuous functions at x which are holomorphic on V. The fundamental result is that X is such a space, and that $X - V$ is an analytic subvariety, if the following conditions are fulfilled: (i) each point $x \in X - V$ has arbitrarily small open neighborhoods U such that $U \cap V$ is an irreducible analytic space; (ii) the local rings A_x induce the structure of an analytic space (of lower dimension than V) on the set $X - V$; (iii) each point $x \in X - V$ has an open neighborhood U such that the continuous functions on U which are holomorphic on V separate points in $U \cap V$. This result was first proved, though with stronger hypotheses, by Baily [Amer. J. Math. **80** (1958), 348–364; MR **20** #5890].

The next lectures turn to the central topic of this portion of the seminar: the analytic compactification of S_n/Γ_n, the quotient of the Siegel space S_n by the symplectic modular group Γ_n; this is a problem first discussed by Siegel [Comm. Pure Appl. Math. **8** (1955), 677–681; MR **17**, 602], and treated in more detail by Satake [J. Indian Math. Soc. (N.S.) **20** (1956), 259–281; MR **18**, 934]. Roughly speaking the compactification is of the form

$$(S_n/\Gamma_n)^* = (S_n/\Gamma_n) \cup (S_{n-1}/\Gamma_{n-1}) \cup \cdots \cup (S_0/\Gamma_0),$$

with a suitable topological and analytical structure. Lecture 12 is devoted to the construction of this compactification as a compact topological space containing S_n/Γ_n as an everywhere-dense open subset; indeed two versions of this construction are discussed, one by Satake and the other by Cartan. Both proceed by considering an auxiliary space $S_n{}^*$ on which Γ_n acts so that $S_n{}^*/\Gamma_n$ is the desired compactification; Satake constructs $S_n{}^*$ abstractly, while Cartan obtains $S_n{}^*$ from the closure of S_n (in the representation of S_n as a bounded domain). In lecture 13 Satake extends his presentation to cover the corresponding problem for discrete subgroups of the symplectic group commensurable with the modular group Γ_n. The next step is the introduction of a complex structure on the compactification. In preparation for this Satake introduces in lecture 14 an extension of the operator $\Phi_{n,r}$ mentioned earlier to local modular forms, that is, to modular forms defined only on Γ_n-invariant open subsets of S_n. By use of this operator and the theory of continuation of complex analytic spaces developed in lecture 11, Cartan and Satake prove in lecture 15 that $(S_n/\Gamma_n)^*$ is a normal complex analytic space, containing $(S_n/\Gamma_n)^* - (S_n/\Gamma_n)$ as a complex analytic subvariety of lower dimension; the corresponding result is also proved for groups commensurable with Γ_n. The final step is the projective imbedding of $(S_n/\Gamma_n)^*$. Again in preparation Satake discusses in lecture 16 conditions under which the mapping $\Phi_{n,r}$ is onto; this is a question originally investigated by Maass [Math. Ann. **123** (1951), 125–151; MR **13**, 210]. Finally in lecture 17 Cartan proves that the ordinary scalar modular forms of sufficiently high weight yield an imbedding of $(S_n/\Gamma_n)^*$ as a normal algebraic subvariety of a projective space, with $(S_n/\Gamma_n) \subset (S_n/\Gamma_n)^*$ a Zariski-open subset; this result is also proved for groups commensurable with Γ_n.

(E) Shimura's lectures. Lecture 18 is devoted to a review of some algebraic-geometric properties of abelian varieties, in particular those related to the notions of a polarized abelian variety, of a projective family of projective varieties, of the field of moduli of a polarized abelian variety, and of Kummer varieties; the references given for other discussions of these concepts are to Matsusaka [Amer. J. Math. **80** (1958), 45–82; MR **20** #878] and Weil [Proc. Internat. Sympos. Algebraic Number Theory, Tokyo and Nikko, 1955, pp. 9–22, Science Council of Japan, Tokyo, 1956; MR **18**, 673]. After a preliminary discussion of the concept of a generic point for holomorphic functions, lecture 19 is concerned

with analytic systems of algebraic varieties. To define such a system suppose that $f_i(u, s)$ $(0 \leq i \leq N)$ are holomorphic functions on a product domain $U \times S \subset C_n \times C_m$ such that: (i) the $(n+1) \times (N+1)$ matrix, whose ith column consists of the function $f_i(u, s)$ and its n first order partial derivatives $\partial f_i / \partial u_j$, has rank $n+1$ everywhere on $U \times S$; (ii) for each $s \in S$ the point set

$$\{(f_0(u, s), \cdots, f_N(u, s)) | u \in U\},$$

considered as lying in a projective space P_N, is a Zariski-open subset of an algebraic variety $V(s) \subset P_N$; (iii) the degree of $V(s)$ is independent of s. The varieties $\{V(s)\}$ then compose an analytic system of algebraic varieties. The principal result is that there are meromorphic functions $\psi_1(s), \cdots, \psi_k(s)$ on S such that $(1, \psi_1(s), \cdots, \psi_k(s))$ is the Chow point of $V(s)$ whenever the functions $\psi_i(s)$ are defined. This is applied to abelian varieties in lecture 20. If δ is a diagonal matrix with entries $\delta_1 = 1$, $\delta_i | \delta_{i+1}$ $(i = 1, \cdots, n-1)$, and z is any point of the Siegel space S_n, the columns of the matrix (z, δ) generate a lattice $D_\delta(z)$ in C_n such that the quotient space $C_n / D_\delta(z)$ is an abelian variety with a canonical polarization associated to the Riemann form $E = \begin{pmatrix} 0 & -\delta \\ \delta & 0 \end{pmatrix}$. By explicitly constructing a family of theta functions satisfying the three conditions listed above these abelian varieties are associated to an analytic system $A_\delta(z)$ of projective abelian varieties. Rather than these varieties, however, consider the projective family $F_\delta(z)$ associated to each $A_\delta(z)$. It is proved that the $F_\delta(z)$ also form an analytic system of algebraic varieties for all $z \in S_n - Y$, where Y is an analytic sub-variety of codimension 1; thus the Chow point of $F_\delta(z)$ can be given in the form $(1, \chi_1(z), \cdots, \chi_\lambda(z))$ for some meromorphic functions $\chi_i(z)$ on S_n. It is then demonstrated that $F_\delta(z) = F_\delta(z^1)$ if and only if $z^1 = Mz$ for some symplectic transformation M in the paramodular group Γ_δ; consequently the functions $\chi_i(z)$ are invariant under the paramodular group. If K_δ denotes the field of moduli and M_δ the field of invariant meromorphic functions under the paramodular group, then also $M_\delta = C \cdot K_\delta$. Somewhat similar results are derived for the Kummer varieties. *R. C. Gunning* (Princeton, N.J.)

2751:

Gunning, R. C. Factors of automorphy and other formal cohomology groups for Lie groups. Ann. of Math. (2) **69** (1959), 314–326; erratum, 734.

La classe des facteurs d'automorphie correspond sur une variété analytique complexe M aux éléments du premier groupe de cohomologie à coefficients dans le groupe multiplicatif des fonctions holomorphes, non nulles sur M. On étend ce point de vue d'une manière formelle en considérant des groupes de coefficients plus généraux: formes différentielles C^∞, formes d ou d'' fermées. Les hypothèses faites sur la variété M sont: (i) l'existence d'un groupe G de Lie, d'automorphismes C^∞, l'existence d'un recouvrement U_i avec des sous-ensembles distingués $V_i \subset U_i$; $gV_i \subset U_i$; $gV_i = V_i$ ou bien $gV_i \cap V_i = \emptyset$; (ii) les $g \in G$ tels que $gV_i = V_i$ forment un sous-groupe G_i compact; (iii) chaque $x_0 \in U_i$ a un voisinage U' qui possède une représentation C^∞ dans G, soit $\tilde{g}_{x_0}$, telle que pour tout $x \in U'$ on ait $x \in \tilde{g}_{x_0} x V_i$; (iv) il existe une partition $e_i(x)$, C^∞, subordonnée au recouvrement U_i, avec $e_i(gx) = e_i(x)$ pour tout $g \in G$. Sous ces hypothèses on a $H^p(G, \psi^r) = 0$, pour tout $p > 0$, $r \geq 0$, et de plus, dans le cas analytique complexe, $H^p(G, \psi^{s,t}) = 0$ pour $p \geq 0$, et $s, t \geq 0$; ψ^r est l'espace des formes C^∞, homogènes de degré r, $\psi^{s,t}$ désigne les formes analogues de bidegré (s, t) dans le cas complexe. Si, de plus, d (*et* d'' dans le cas complexe) déterminent des homomorphismes surjectifs, on a $H^p(G, \varphi^r) \cong \mathscr{H}^{p+r}(M/G)$ pour $p > 0$, $r \geq 0$, et, dans le cas complexe: $H^p(G, \varphi^{s,t}) \cong \mathscr{H}^{s,p+t}(M/G)$. φ^r est le noyau de l'application par d: $\psi^r \to \psi^{r+1}$, et $\mathscr{H}^r(M/G) \cong H^0(G, \varphi^r)/dH^0(G, \psi^{r-1})$. Si G est un groupe proprement discontinu d'automorphismes analytiques de M on retrouve les groupes de de Rham et de P. Dolbeault de M/G. Soit A le groupe des fonctions holomorphes sur M, A^* le groupe multiplicatif de celles qui ne s'annulent pas sur M; on étudie les suites exactes formées avec les $H^p(G, A)$, $H^p(G, A^*)$, $H^{p+1}(G, Z)$. Si G est connexe et M simplement connexe, $H^p(G, A^*) \cong H^p(G, A)$ pour $p \geq 3$. On a d'autres suites exactes en introduisant les $\mathrm{Hom}[\pi(G), Z]$. Des applications de ce formalisme sont données dans le cas où M est un domaine symétrique de E. Cartan. *P. Lelong* (Paris)

2752:

Boothby, William M. Hermitian manifolds with zero curvature. Michigan Math. J. **5** (1958), 229–233.

The main theorem of this paper is the following: If M is a compact Hermitian manifold with curvature zero, then its universal covering space $\tilde{M}$ is a complex Lie group, and M is the factor space of $\tilde{M}$ by a discontinuous group D of transformations each of which is an isometry without fixed points of $\tilde{M}$. It is also pointed out that D need not be a subgroup of $\tilde{M}$. The proof is achieved by slightly generalizing the lemma of Bochner. A complex manifold of complex dimension n is said to be parallelisable if there exist n analytic vector fields defined over it which are linearly independent at each point. It is proved also that a simply connected complex manifold is a Hermitian manifold of zero curvature if and only if it is parallelisable. Moreover, a parallelisable complex manifold has a natural Hermitian metric, thus showing that the above main theorem is a generalization of a theorem of H. C. Wang, though in the latter case D is a subgroup of $\tilde{M}$. *M. Kuranishi* (Nagoya)

SPECIAL FUNCTIONS

See also 2847, 3102.

2753:

Graesser, R. F. An infinite set of formulas connecting binomial coefficients. Math. Mag. **32** (1958/59), 153–154.

Using the identity

$$(1 + x + \cdots + x^n)^t = (1 - x^{n+1})^t (1 - x)^{-t},$$

the author obtains

$$\sum_{r=0}^{R} (-1)^r \binom{i}{r} \left((i - r) \binom{n+1}{i-1} + t - 1 \right) = \begin{cases} 1 & (t = 0), \\ 0 & (t \geq 1), \end{cases}$$

where

$$R = \min\left(i, \left[\frac{ni + t}{n+1}\right]\right)$$

[compare E. Netto, *Lehrbuch der Combinatorik*, Teubner, Leipzig, 1927; p. 255, formula (441)]. *L. Carlitz* (Durham, N.C.)

2754:

Carlitz, L. Note on the coefficients of cosh x/cos x. Math. Mag. **32** (1958/59), 132, 136.

In a recent paper [same Mag. **31** (1957/58), 185–191; MR **20** #5301] J. M. Gandhi defined a set of rational integers S_{2n} by means of

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} S_{2n} \frac{x^{2n}}{(2n)!}$$

and conjectured that $S_{2n} = 2^n S_{2n}'$, where S_{2n}' is odd. In this note it is shown that the conjecture is correct and moreover that $S_{2n}' \equiv (-1)^{n(n-1)/2} \pmod 4$. The proof uses mathematical induction.

A. L. Whiteman (Princeton, N.J.)

2755:

Kelisky, R. P. Inverse elliptic functions and Legendre polynomials. Amer. Math. Monthly **66** (1959), 480–483.

The author derives series expansions for complete elliptic integrals $K(k)$, $E(k)$ in terms of Legendre polynomials with argument greater than one.

M. D. Friedman (Needham Heights, Mass.)

2756:

Schottlaender, Stefan. Reihen nach Hankelschen Funktionen. Math. Nachr. **19** (1958), 211–225.

Let $\mathrm{Im}(\delta) \geqq 0$, $\mathrm{Im}(a) \geqq 0$, $\mathrm{Im}(y) > 0$, and $\mathrm{Im}(\delta + a) > 0$. First, the author shows that

$$\sum_{n=0}^{\infty} e^{ins} H_\nu^{(1)}(y+na) = (-i)^{\nu+1} \frac{2}{\pi} \int_0^{\infty} \frac{\exp(iy \cosh t)}{1 - \exp[i(\delta + a \cosh t)]} \cosh \nu t \, dt.$$

Secondly, for purposes of practical calculations, the integral is approximated by elementary functions. This approximation is as accurate as desired; bounds for the error terms are derived. No numerical details.

C. J. Bouwkamp (Eindhoven)

2757:

Sips, Robert. Représentation asymptotique des fonctions de Mathieu et des fonctions sphéroidales. II. Trans. Amer. Math. Soc. **90** (1959), 340–368.

In this paper, asymptotic formulas are given for Mathieu and spheroidal functions in regions where previously deduced formulas [same Trans. **66** (1949), 93–134; MR **11**, 435] are useless. The author starts from these previous formulas for Mathieu functions, which are remarkably accurate if the independent variable is in the neighbourhood of $\pi/2$. He then uses an infinite integral representation, which leads to a power series from which new formulas for the eigenvalues follow. These are then used to obtain the required asymptotic values for the Mathieu functions. A similar procedure is applied to the elongated and to the flattened spheroidal wave functions. The calculations are carried out very fully, and numerical values and comparisons with known formulas are given.

M. J. O. Strutt (Zurich)

2758:

Meligy, A. S. Expansions for the logarithmic-integral and cosine-integral functions. Proc. Roy. Irish Acad. Sect. A **59** (1958), 25–28.

Expansions of li(z) and Ci(z) in terms of

$$(\ln z)^r [I_r(\tfrac{1}{2} \ln z) - I_{r+1}(\tfrac{1}{2} \ln z)]$$

and

$$z^r [(\sin \tfrac{1}{2}z) J_r(\tfrac{1}{2}z) + (\cos \tfrac{1}{2}z) J_{r+1}(\tfrac{1}{2}z)],$$

respectively, are given.

N. D. Kazarinoff (Madison, Wis.)

2759:

Bhonsle, B. R. On some results involving Jacobi polynomials. Bull. Calcutta Math. Soc. **50** (1958), 159–164.

Using a result of Brafman [Proc. Amer. Math. Soc. **2** (1951), 942–949; MR **13**, 649] the author proves that if $|x| > 1$,

$$F(x) Q_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(x-1)^{-\alpha}(x+1)^{-\beta} \int_{-1}^{+1} \frac{(1-t)^\alpha (1+t)^\beta F(t) P_n^{(\alpha,\beta)}(t) dt}{x-t},$$

where $F(t)$ is a polynomial of degree $\leqq n$. Several special cases and consequences are considered and misprints in two formulas of Carlitz [Bull. Calcutta Math. Soc. **46** (1954), 93–95; MR **16**, 694] for $d^r/dx^r [P_n^{(\alpha,\beta)}(\pm 1)]$ are corrected.

N. D. Kazarinoff (Madison, Wis.)

2760:

Videnskiĭ, V. S. Stieltjes and Bernstein's inequalities for Legendre polynomials. Dokl. Akad. Nauk SSSR **124** (1959), 973–975. (Russian)

If a polynomial $R_n(x)$ of degree $\leqq n$ satisfies

$$(1-x^2)^{1/2} |R_n(x)| \leqq L, \quad -1 \leqq x \leqq 1$$

for a given positive number L then

$$(1-x^2) |R_n'(x)| \leqq 4L(n+1), \quad -1 \leqq x \leqq 1.$$

On the basis of this theorem the author shows how certain of the Bernstein inequalities for Legendre polynomials follow, in form at least, from the others. Moreover, the method indicates that such inequalities are not peculiar to Legendre polynomials but are satisfied in form by any polynomial.

R. G. Langebartel (Urbana, Ill.)

2761:

Carlitz, L. Bernoulli and Euler numbers and orthogonal polynomials. Duke Math. J. **26** (1959), 1–15.

The author generalizes a result of Touchard [Canad. J. Math. **8** (1956), 305–320; MR **18**, 16] concerning polynomials symbolically orthogonal with respect to the Bernoulli numbers. He constructs a set of polynomials $\Omega_n^{(\lambda)}(z)$ such that with the usual symbolic notation

$$(1) \qquad \beta^r \Omega_n^{(\lambda)}(\beta) = K_n^{(\lambda)} \delta_{rn} \quad (0 \leqq r \leqq n),$$

where $\Omega_n^{(\lambda)}(z) = (-2)^n (\lambda+1)_n \binom{2n}{n}^{-1} F_n^\lambda(1-\lambda+2z)$,

$F_n^m(z)$ is the generalized polynomial of Pasternak, and

$$K_n^{(\lambda)} = \frac{(-1)^n}{2n+1} \frac{1}{2^n} \frac{(n!)^2 (1+\lambda)_n (1-\lambda)_n}{\{1 \cdot 3 \cdot 5 \cdots (2n-1)\}^2}.$$

$B_n(\lambda)$ is the Bernoulli polynomial defined by

$$ze^{\lambda z}(e^z - 1)^{-1} = \sum_{n=0}^{\infty} B_n(\lambda) z^n / n!,$$

and $\beta_n = \beta_n(\lambda) = [B_{n+1}(\lambda) - B_{n+1}]/(n+1)\lambda$.

For $\lambda=0$, $\beta_n(0)=B_n$ and (1) reduces to Touchard's result. Carlitz writes $\Omega_n(z)$ in place of Touchard's $Q_n(z)$.

If the "Eulerian" numbers are defined by

$$\frac{1-\lambda}{e^z-\lambda}=\sum_{n=0}^{\infty} R_n(\lambda)x^n/n!, \quad \lambda\neq 1,$$

then the author shows that the polynomials $A_n^{(\lambda)}(x)$ defined by

$$(1+\lambda t)^x/(1-t)^{x+1}=\sum_{n=0}^{\infty} A_n^{(\lambda)}(x)t^n$$

are symbolically orthogonal with respect to the Eulerian numbers by the symbolic relation (2) $R^r(\lambda)A_n^{(\lambda)}(R(\lambda))=n!\lambda^n\delta_{rn}/(1-\lambda)^n$ $(0\leq r\leq n)$. If $\lambda=-1$, (2) includes as a special case a symbolic orthogonality relation for the number $C_n=2^nE_n(0)$, where $E_n(x)$ is the Euler polynomial. If the polynomial $f_n(u)$ is defined by $n!A_n(x)=f_n(2x+1)$ then it is shown that these polynomials satisfy a symbolic orthogonality relation with respect to the Euler numbers. These results are generalized to Euler numbers of higher order.

The author shows that the polynomials $\Omega_n^{(\lambda)}(z)$ and $A_n^{(\lambda)}(z)$ satisfy true orthogonality relations. For $\Omega_n^{(\lambda)}(z)$ the relation is

$$(3)\quad \frac{\sin\pi\lambda}{2i\lambda}\int_{\alpha-i\infty}^{\alpha+i\infty}\Omega_m^{(\lambda)}(z)\Omega_n^{(\lambda)}(z)\frac{dz}{\sin\pi z\sin\pi(z-\lambda)}=2^nK_n^{(\lambda)}\delta_{mn},$$

where $-1<\alpha<0$ and $0<\lambda<1$. If $\lambda\to 0$ then (3) implies the orthogonality relation satisfied by Touchard's polynomials $\Omega_n(z)$. For $A_n^{(\lambda)}(z)$ there are two orthogonality relations, namely

$$\frac{1-\lambda}{2i\lambda}\int_{\alpha-i\infty}^{\alpha+i\infty}A_m^{(\lambda)}(z)A_n^{(\lambda)}(z)\frac{(-\lambda)^{-z}dz}{\sin\pi z}=\lambda^n\delta_{mn} \qquad (-1<\alpha<0),$$

and

$$\sum_{j=0}^{\infty}A_m^{(\lambda)}(j)A_n^{(\lambda)}(j)\lambda^{-j-1}=\frac{\lambda^n}{\lambda-1}\delta_{mn} \quad (|\lambda|>1).$$

R. P. Kelisky (Yorktown Heights, N.Y.)

2762:

Dickinson, David. On certain polynomials associated with orthogonal polynomials. Boll. Un. Mat. Ital. (3) **13** (1958), 116–124.

Der Verfasser betrachtet Polynomsysteme, die der klassischen Rekursionsformel $p_n(x)-(a_n+b_nx)p_{n-1}(x)+c_np_{n-2}(x)=0$ genügen, wobei die klassische Bedingung $a_na_{n+1}c_n>0$ durch die Bedingung $a_nc_n\neq 0$ ersetzt wird, und zeigt, dass solche Polynomsysteme, zusammen mit den sog. assoziierten Polynomen in gewisser Weise orthogonal sind. Als Beispiel und Anwendung werden die Legendreschen Polynome untersucht. *K. Endl* (Columbus, Ohio)

2763:

Weston, V. H. Toroidal wave functions. Quart. Appl. Math. **16** (1958), 237–257.

In toroidal coordinates

$$x=\frac{d\sinh\xi\cos\phi}{\cosh\xi-\cos\eta},\quad y=\frac{d\sinh\xi\sin\phi}{\cosh\xi-\cos\eta},\quad z=\frac{d\sin\eta}{\cosh\xi-\cos\eta},$$

the Helmholtz equation takes a form in which only the variable ϕ can be separated out. The author applies the method of his paper in same Quart. **15** (1957), 420–427 [MR **20** #4700] to obtain a solution of the resulting equation in ξ, η in the form of a power series in $(\cosh\xi-\cos\eta)^{-1}$, the coefficients involving associated Legendre functions of $\cosh\xi$. Integral representations and asymptotic behaviour are studied, leading up to the complete set of solutions representing radiation from a ring source. *F. M. Arscott* (London)

ORDINARY DIFFERENTIAL EQUATIONS

See also 2741, 2743, 2805, 2916, 3122, 3123.

2764:

Plehotin, A. P. Theorem on the existence and uniqueness of the solution of a boundary value problem for a system of ordinary differential equations. Dokl. Akad. Nauk SSSR **123** (1958), 613–615. (Russian)

The system (1) $y'=f(t,y)$ is considered for n-vectors y, together with a boundary condition

$$(2)\qquad \sum_{m=0}^{\mu}\alpha_m y(t_m)=b, \quad t_0\leq t_1\leq\cdots\leq t_\mu.$$

Here the α_m are constant n by n matrices, and b is a constant n-vector. The author states a set of rather complicated conditions which will guarantee the existence of a unique solution of (1), (2) for $t_0\leq t\leq t_\mu$, which moreover satisfies a certain estimate in terms of an approximate solution. *E. A. Coddington* (Los Angeles, Calif.)

2765:

Abian, Smbat; and Brown, Arthur B. On the solution of the differential equation $f(x,y,y')=0$. Amer. Math. Monthly **66** (1959), 192–199.

The authors establish local existence and uniqueness of a solution of $f(x,y,y')=0$, $y(x_0)=y_0$, under conditions weaker than those usually assumed. A value z_0 is chosen so that $f(x_0,y_0,z_0)$ is small, and the following assumptions are made about $f(x,y,z)$ in a neighborhood of (x_0,y_0,z_0). f is assumed to be continuous and to satisfy a Lipschitz condition in y. Positive constants D_2 and D_3 are assumed to exist such that if $z_1\neq z_2$,

$$D_2\leq(f(x,y,z_2)-f(x,y,z_1))/(z_2-z_1)\leq D_3.$$

The proof is by successive approximations, applied to $y'=F(x,y,y')$, where $F(x,y,y')=y'-kf(x,y,y')$ for some k chosen sufficiently small. The initial approximation $Y_1(x)$ is given by $Y_1(x)=y_0+\int_{z_0}^{z}F(t,y_0,z_0)dt$. If $Y(x)$ is the solution, local estimates on $|Y(x)-Y_n(x)|$ are obtained which converge to zero as the nth power of a small quantity. The theorem is illustrated with an example which shows how the choice of z_0 can possibly determine to which of several solutions the process may converge.

W. S. Loud (Minneapolis, Minn.)

2766:

Abian, Smbat; and Brown, Arthur B. On the solution of the differential equation $f(x,y,y^{(1)},\cdots,y^{(n)})=0$. Boll. Un. Mat. Ital. (3) **13** (1958), 383–394. (Italian summary)

This paper shows how to apply the method of successive

substitutions directly to equations of the form given in the title, without solving for the nth derivative. [Compare the previous paper of the authors, reviewed above, for the case $n=1$.] It is not assumed that any partial derivatives of the function f exist. Four estimates (inequalities (53), (54), (55) and (56)) of the error of the mth approximation are given. (55) and (56) are corollaries of (53) and (54) respectively, but would seem likely to give smaller estimates of the error when $m>2$. However, (53) and (54) might be useful at the outset of a computation to place a bound on the number of successive substitutions required.

L. M. Graves (Chicago, Ill.)

2767:

Abian, Smbat; and Brown, Arthur B. On the solution of simultaneous first order implicit differential equations. Math. Ann. **137** (1959), 9–16.

The authors prove by successive approximations an existence and a uniqueness theorem for the system

$$F_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \quad (i = 1, \cdots, n).$$

The method does not require solving for the derivatives. Four appraisals of the error are given, two of which are independent of previous computational errors. {It appears that lemma 4 does not in fact apply to (47) unless $|c_i| \leq V_i$, since $U_i(x)=b_i$ implies $U_i'(x)=0$.}

W. J. Coles (Salt Lake City, Utah)

2768:

Stokes, Arnold. The application of a fixed-point theorem to a variety of nonlinear stability problems. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 231–235.

The author employs the following fixed point theorem, a consequence of a theorem of Tychonoff [Math. Ann. **111** (1935), 767–776]. Theorem: Let T be a continuous compact mapping of a complete, locally convex linear space E into itself. If A is a closed, convex, bounded subset of E, and $T(A) \subset A$, then there exists a fixed point of T in E. The author treats the vector systems (1) $\dot{x}=f(t, x)$, and (2) $\dot{x}=A(t)x+f(t, x)$, the latter in the cases that all solutions of (3) $\dot{x}=A(t)x$ are bounded, and that all solutions of (3) are exponentially decreasing. He obtains theorems on existence in the large, boundedness, and ultimate boundedness of solutions of (1) and (2). Also, assuming $f(t, 0)=0$, he obtains criteria for stability and asymptotic stability of $x=0$.

The technique is to use for E the space of real continuous n-vector functions of t on $t \geq 0$. The set A is those $x \in E$ satisfying an inequality $\|x(t)\| \leq g(t)$, with $g(t)$ positive and continuous for $t \geq 0$. The mapping T is obtained by converting the differential equations to integral equations. The function $g(t)$ is found as a solution of a differential inequality or differential equation determined by bounds on $f(t, x)$. The results are not new. Results due to Coddington and Levinson, Wintner, Bellman, and Perron are deduced as corollaries. A typical result is that if $\|f(t, x)\| \leq G(t, \|x\|)$, and if $g(t)$ can be found satisfying $\dot{g}(t) \geq G(t, g(t))$, $g(0)=r_0$, for any $r_0>0$ and for all $t \geq 0$, then (1) has a solution for all $t \geq 0$ satisfying $x(0)=b$ for any n-vector b.

W. S. Loud (Madison, Wis.)

2769:

Rachajsky, B. Principe de l'intégration des équations différentielles ordinaires à l'aide de la différentiation et la théorie des transformations de contact. Applications. Bull. Soc. Math. Phys. Serbie **8** (1956), 157–164. (Serbo-Croatian. French summary)

In this article the author takes an ordinary differential equation $A(x, y, p)=0$, and shows in this form that the possibility of integration "by means of differentiation" (as in the case of the equation of Lagrange) is equivalent to the existence of a contact transformation, reducing the given equation to the form $a(x_1, y_1)=0$.

P. Musen (Cincinnati, Ohio)

2770:

Kimura, Tosihusa. Sur les fonctions définies par les équations différentielles ordinaires du premier ordre. Comment. Math. Univ. St. Paul. **7** (1959), 65–75.

The author continues his previous work [same Comment. **2** (1953), 23–28; **3** (1954), 43–49; **3** (1955), 97–107; **4** (1955), 25–41; MR **15**, 311; **16**, 700, 820; **17**, 364] on generalizations of Malmquist's theorem [Acta Math. **36** (1912/13), 297–343]. He considers the differential equation (1) $y'=f(x, y)$ (x, y complex), where f is uniform and meromorphic in a domain D of the x, y space. If (x_0, y_0) denotes a singular point of $f(x, y)$ and if $g(x)$ denotes an integral of (1), then several results are proved concerning the behavior of $g(x)$ near the singularity x_0. These results require too many preliminary definitions to be stated here, but are similar to those published by the author previously [loc. cit.].

In conclusion the author proves results giving conditions on D such that each integral $g(x)$ of (1) has the property of Iversen—namely, $g(x)$ has at least one functional element with center in D; and if $P(x, a)$ is such an element centered at a, and L a segment in D connecting a to b, then P can be continued as close as desired to b.

C. S. Coleman (Claremont, Calif.)

2771:

Fer, F. Résolution de l'équation matricielle $dU/dt=pU$ par produit infini d'exponentielles matricielles. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 818–829.

The author shows that the solution of the matrix differential equation $dU/dt=pU$ can be written in the form $U=e^{P}e^{P_1}e^{P_2}\cdots$, where $P=\int_0^t p\,dt$, $P_n=\int_0^t p_n dt$, and

$$p_n = e^{-P_{n-1}}p_{n-1}e^{P_{n-1}} + \int_0^{-1} e^{sP_{n-1}}p_{n-1}e^{-sP_{n-1}}ds.$$

R. Bellman (Santa Monica, Calif.)

2772:

Cordes, Heinz Otto. Über die Eigenwertdichte von Sturm-Liouville-Problemen am unteren Rand des Spektrums. Math. Nachr. **19** (1958), 64–72.

Let $q_1(x)$ and $q_2(x)$ be continuous on the interval $a \leq x \leq b$, and consider the two boundary value problems (B_1): $L_1u \equiv -u''+q_1(x)u=\lambda u$ with $u(a)=u(b)=0$ and (B_2): $L_2u \equiv y^{(4)}-(q_2(x)u')'=\lambda u$ with $u(a)=u'(a)=u(b)=u'(b)=0$. Let $\varepsilon>0$ and the positive integer N be arbitrary. It is shown for $k=1, 2$ that there exist functions $q_k(x)$ such that the first N eigenvalues $\lambda_1^k, \lambda_2^k, \cdots, \lambda_N^k$ (counting multiplicities) of each of the two boundary value problems (B_k) lie in the interval $\lambda_1^k \leq \lambda \leq \lambda_1^k+\varepsilon$.

C. R. Putnam (Lafayette, Ind.)

2773:

Kabanova, V. On the expansion in characteristic vector-functions of non-selfadjoint second order differential

systems. Dokl. Akad. Nauk SSSR **121** (1958), 30–33. (Russian)

The equation (*) $Y''-YP(x)+\lambda^2 Y=0$ for $m\times m$ matrix-valued functions Y on $0\leq x<\infty$ is considered. Here P is a square matrix whose components are complex-valued functions. The author states results (without proofs) which are extensions of earlier work of B. Ya. Levin [same Dokl. **106** (1956), 187–190; MR **18**, 35] on the scalar equation of the same type. The fundamental solution of (*) is that solution Y satisfying $|Y(x,\lambda)-e^{i\lambda x}I|\to 0$, $x\to\infty$, where I is the identity matrix. If $\int_0^\infty x|P(x)|dx<\infty$ there exists a fundamental solution of (*) which has the representation

$$Y(x,\lambda)=e^{i\lambda x}I+\int_x^\infty K(x,t)e^{i\lambda t}dt \quad (\operatorname{Im}\lambda\geq 0),$$

where K is continuous for $0<x\leq t<\infty$ and

$$\int_0^\infty\int_x^\infty |K(x,t)|^2dtdx<\infty, \quad \int_x^\infty |K(x,t)|dt<\infty \quad (x\geq 0).$$

The resolvent kernel R for the integral equation

$$f(x)+\int_0^x K(s,x)f(s)ds=g(x)$$

is continuous for $0<s\leq x<\infty$ and $\int_0^\infty\int_0^x |R(s,x)|^2dsdx<\infty$. Let $L_m^2(0,\infty)$ be the space of m-vector functions whose components are in $L^2(0,\infty)$ and which are zero on the negative half-axis. Let $L_m^{2(+)}$ be the space of m-vector functions which are holomorphic and bounded in the upper half-plane, and which are in L^2 on the real line. The mapping

$$\phi(\lambda)=\frac{1}{\sqrt{(2\pi)}}\text{l.i.m.}\int_0^\infty Y(x,\lambda)f(x)dx,$$

where Y is a fundamental solution, takes $L_m^2(0,\infty)$ into $L_m^{2(+)}$, and

$$f(x)=\frac{1}{\sqrt{(2\pi)}}\text{l.i.m.}\int_{-\infty}^\infty Z(x,\lambda)\phi(\lambda)d\lambda$$

takes $L_m^{2(+)}$ into $L_m^2(0,\infty)$. Here

$$Z(x,\lambda)=e^{-i\lambda x}I+\int_0^x R(s,x)e^{-i\lambda s}ds.$$

Moreover, $c_1\|\phi\|\leq\|f\|\leq c_2\|\phi\|$, where c_1, c_2 are positive constants. The matrix Z is not in general a solution of (*). However, it can be shown that Z satisfies

$$Z''-P(x)Z+\lambda^2 Z=-i\lambda R(0,x)+\frac{\partial R}{\partial s}(0,x).$$

The expansion result in terms of the fundamental solution Y of (*) is given by

$$f(x)=\frac{1}{\sqrt{(2\pi)}}\text{l.i.m.}\int_{-\infty}^\infty Y(x,\lambda)\psi(\lambda)d\lambda,$$

where

$$\psi(\lambda)=\frac{1}{\sqrt{(2\pi)}}\text{l.i.m.}\int_0^\infty Z(x,\lambda)f(x)dx.$$

Further results of a similar nature are given for the case when P satisfies $\int_0^\infty (1+x)|P(x)|dx<\infty$.

E. A. Coddington (Los Angeles, Calif.)

2774:

Berezanskiĭ, Yu. M. Representation of positive definite kernels by eigenfunctions of differential equations. Mat. Sb. (N.S.) **47 (89)** (1959), 145–176. (Russian)

Proofs of results announced earlier [Dokl. Akad. Nauk SSSR **110** (1956), 893–896; MR **19**, 1061].

L. Gårding (Lund)

2775:

Kac, I. S. Some general theorems on the behaviour of spectral functions of second order differential systems. Dokl. Akad. Nauk SSSR **122** (1958), 974–977. (Russian)

Two theorems are stated concerning the asymptotic behavior of the spectral functions for the problem (1) consisting of the equation $-(py')'+qy-\lambda\rho y=0$ on $0\leq x<L\leq\infty$ and the boundary condition $y(0)=n$, $(py')(0)=m$. Here m, n $(m^2+n^2>0)$ are real constants, λ is a complex parameter, $\rho(x)\geq 0$, $p(x)>0$, and q are real measurable functions such that for any l, $0<l<L$,

$$0<\int_0^l \rho(x)dx<\infty, \quad \int_0^l dx/p(x)<\infty, \quad \int_0^l |q(x)|dx<\infty.$$

Let $L^2(M)$ denote the L^2-space with the measure generated by $M(x)=\int_0^x \rho(s)ds$. A function τ on $-\infty<\lambda<\infty$, which is left continuous, is a spectral function for the problem if the Parseval equality

$$\int_{-\infty}^\infty \left|\int_0^L f(x)u(x,\lambda)dM(x)\right|^2 d\tau(\lambda)=\int_0^L |f(x)|^2dM(x)$$

is valid for all $f\in L^2(M)$ which vanish outside compact subsets of $(0,\infty)$. Here u is a solution of the differential equation.

A non-decreasing function ω on $1\leq\lambda<\infty$ is said to be of class K_ν if $\omega(\lambda)\to\infty$ $(\lambda\to\infty)$ and there exist numbers $\gamma<\nu$ and a sufficiently large $N>1$ such that for $\eta>\lambda>N$, $\omega(\eta)/\omega(\lambda)<(\eta/\lambda)^\gamma$. A non-decreasing θ on $1\leq\lambda<\infty$ is of class $\bar{K}_\nu$ if there exists a function $\omega\in K_\nu$ such that $\theta(\lambda)/\omega(\lambda)\to 1$ $(\lambda\to\infty)$.

Together with the original problem is considered a similar problem (2) with p, q, ρ, L, n, m replaced by p_0, q_0, ρ_0, L_0, n_0, m_0, respectively. Theorem 1 states that if $n=n_0\neq 0$, $p(x)/p_0(x)\to 1$ $(x\to\infty)$, $\rho(x)/\rho_0(x)\to 1$ $(x\to 0)$, and at least one of the spectral functions τ_0 for (2) is in $\bar{K}_1$, then for any spectral function τ of (1) we have $\tau(\lambda)/\tau_0(\lambda)\to 1$ $(\lambda\to\infty)$. Theorem 2 states that if $n=n_0=0$, $m=m_0\neq 0$, if at least one spectral function τ_0 of (2) is in K_2, and if σ_0 defined by $\sigma_0(\lambda)=\int_1^\lambda \xi^{-1}d\tau_0(\xi)$ $(\lambda>1)$ is in $\bar{K}_1$, $\lambda\tau_0^{-1}(\lambda)\sigma_0(\lambda)\to A<\infty$ $(\lambda\to\infty)$, then any spectral function τ of (1) satisfies $\tau(\lambda)/\tau_0(\lambda)\to 1$ $(\lambda\to\infty)$.

Special cases of these results yield previously known asymptotic estimates for the spectral functions of (1).

E. A. Coddington (Los Angeles, Calif.)

2776:

Kac, I. S. Growth of spectral functions of differential systems of second order. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 257–274. (Russian)

The problem

$$(*) \qquad -y''+qy-\lambda\rho y=0 \quad (0\leq x<L\leq\infty),$$
$$y'(0)=m,\ y(0)=n \quad (\operatorname{Im}m=\operatorname{Im}n=0;\ m^2+n^2\neq 0)$$

is considered. Here ρ $(\rho(x)>0)$ and q are real-valued measurable functions which are summable on any segment $0\leq x\leq l$, $0<l<L$. Let $M(x)=\int_0^x \rho(s)ds$, and let u be the

solution of (*). For $f \in L_0^2(M)$ (the set of all functions in $L^2(M)$ which vanish outside compact subsets of $0 \leq x < L$) let $F(\lambda) = \int_0^L f(x)u(x, \lambda)dM(x)$. A left-continuous non-decreasing function τ on $-\infty < \lambda < \infty$ for which $\tau(0) = 0$, and such that the map $f \to F$ is an isometry of $L_0^2(M)$ into $L^2(\tau)$, is called a spectral function of (*). The following result is proved. If τ is a spectral function of (*), then necessary and sufficient in order that

$$\int_{-\infty}^{\infty} \frac{d\tau(\lambda)}{1+|\lambda|^\alpha} < \infty$$

for some real α is that $\int_0^l (\int_0^x M(s)ds)^{n-1}dx$ converge in case $n \neq 0$, and $\int_0^l (\int_0^x sdM(s))^{n-2}dM(x)$ converge in case $n = 0$, for at least one l, $0 < l < L$.

E. A. Coddington (Los Angeles, Calif.)

2777:

Golomb, Michael. Bounds for solutions of nonlinear differential systems. Arch. Rational Mech. Anal. **1** (1958), 272–282.

Real systems of the form (*) $x' = A(t)x + F(t, x)$ are considered, where x, F are n-vectors, and A is a square matrix function. Bounds for the solutions are obtained in terms of bounds for the general solution of the linear system $x' = A(t)x$. The components of A and F are defined for all $t \geq t_0$ and all real x, and are locally Lebesgue integrable in t for fixed x, continuous in x for almost all $t \geq t_0$. It is assumed that $\|F(t, x)\| \leq a(t)\phi(\|x\|)$, $t \geq t_0$, where a is integrable, and ϕ is a positive non-decreasing continuous function satisfying $\phi(uv) \leq \phi(u)\phi(v)$. The norm used is the sum of the absolute values of the components. Two typical results will be mentioned.

Equation (*) has a solution χ with $\chi(t_0) = x_0 \neq 0$ in any interval $t_0 \leq t \leq T$ for which

$$s(T) = \int_0^T a(\tau)b(\tau)\phi(\|X(\tau)\|)d\tau < \frac{\|x_0\|}{\phi(\|x_0\|)} \int_1^\infty \frac{du}{\phi(u)},$$

and an upper bound is given by

$$\|\chi(t)\| \leq \|x_0\|\psi^*\left(\frac{\phi(\|x_0\|)}{\|x_0\|}s(t)\right)\|X(t)\|, \quad t_0 \leq t \leq T.$$

If $\int_1^\infty du/\phi(u) = \infty$ then the solution exists for all $t \geq t_0$, and any $x_0 \neq 0$. If moreover $s(\infty)$ is convergent, then $\|\chi(t)\|/\|X(t)\|$ is bounded on $t \geq t_0$, and every solution is bounded if the system $x' = A(t)x$ is stable, and tends to 0 as $t \to \infty$ if $x' = A(t)x$ is asymptotically stable. Here X is the fundamental matrix for $x' = A(t)x$ which reduces to the identity matrix at t_0, and $b(t) \leq \|X^{-1}(t)\|$, $t_0 \leq t$. The function ψ^* is the inverse of the function ψ defined by $\psi(x) = \int_1^x du/\phi(u)$, $x > 0$.

If $\int_0^a du/\phi(u) = \infty$ $(a > 0)$ then (*) has a solution χ in the arbitrarily large interval $t_0 \leq t < T$ with upper bound

$$\|\chi(t)\| \leq \omega^*\left(\frac{\omega(\|x_0\|)}{1-\omega(\|x_0\|)s(t)}\right)\|X(t)\|, \quad t_0 \leq t < T,$$

provided x_0 is so small that $s(T) < \int_{\|x_0\|}^R du/\phi(u)$. Here R is any positive number greater than

$$\|x_0\|\omega^*(\phi(\|x_0\|))s(T)/\|x_0\|),$$

or $R = \infty$ if $\int_1^\infty du/\phi(u) < \infty$. The function ω is defined by $\omega(0) = 0$, $\omega(x) = 1/\int_x^R du/\phi(u)$, for $0 < x < R$; whereas ω^* is the inverse of ω defined for $0 \leq y < \infty$.

E. A. Coddington (Los Angeles, Calif.)

2778:

Lykova, O. B. Investigation of the solution of a system of differential equations with a small parameter on a two-dimensional local integral manifold in the "non-resonance" case. Ukrain. Mat. Ž. **10** (1958), 239–250. (Russian. English summary)

Consider the system of equations

$$(*) \qquad dx/dt = X(x) + \varepsilon X^*(t, x, \varepsilon),$$

where ε is a small parameter. The author has proved [same Ž. **9** (1957), 419–431; MR **19**, 857] the existence of a two-dimensional local integral manifold S of (*) in the neighborhood of such a manifold for the unperturbed system $\dot{x} = X(x)$. In the present paper the author studies the structure of individual solutions of (*) on S in the non-resonance case. In particular, the existence of exact stationary solutions of (*) on S is proved and their behavior is described. This result generalizes results of Bogolyubov concerning solutions on a one-dimensional manifold of $\dot{x} = \varepsilon X(t, x)$ reported in Bogolyubov and Mitropol'skiĭ, *Asimptotičeskie metody v teorii nelineĭnyh kolebaniĭ* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **17**, 368]. The proof is based on the methods used by Krylov and Bogolyubov to study the equation $\ddot{x} + w^2x = \varepsilon f(t, x, \dot{x}, \varepsilon)$ in their monograph *Application of the methods of non-linear mechanics to the study of stationary oscillations* [Monogr. Akad. Nauk Ukrain. SSR, No. 8 (1934)].

D. G. Aronson (Minneapolis, Minn.)

2779:

Kuzmak, G. E. The asymptotic solutions of the equations of motion for a dissipative system with one degree of freedom and slowly variable parameters. Dokl. Akad. Nauk SSSR **121** (1958), 37–40. (Russian)

Consider the equation

$$(*) \qquad d^2y/dt^2 + f(\tau, dy/dt) + \varepsilon F(\tau, y) = 0,$$

where $\tau = \varepsilon t$ and ε is a small parameter. Assume that $f(\tau, 0) = 0$, $f_z(\tau, z)|_{z=0} > 0$, f and its first two derivatives with respect to τ are analytic functions of z, and F is analytic in y and continuous in τ. The author states and sketches the proof of the following theorem: If f and F satisfy the above conditions, then for $|\varepsilon| \leq \varepsilon_0$ the functions $y_0(t) = y_0(\tau, w)$ and $(dy/dt)_0 = \phi(\tau)\partial y_0/\partial w$ for $w = \int \phi(\tau)dt$ approximate the solution of (*) and its derivative, respectively, with an error $O(\varepsilon)$ on a time-interval of length $O(1/\varepsilon)$. Here $\phi(\tau) = f_z(\tau, 0)$ and $y_0(\tau, w) = B_0(\tau) + \sum_{n=1}^\infty B_n(\tau)e^{-n[w+c(\tau)]}$, where B_0, c satisfy certain additional conditions.

D. G. Aronson (Minneapolis, Minn.)

2780:

Ehrmann, Hans. Nachweis periodischer Lösungen bei gewissen nichtlinearen Schwingungsdifferentialgleichungen. Arch. Rational Mech. Anal. **1** (1957), 124–138.

In the differential equation (1) $x'' = f(t, x, x')$ $(' = d/dt)$, let f have period T in t. The author proves under appropriate, rather complicated, hypotheses that (1) has infinitely many solutions with period T (but not necessarily least period T). The results are specialized to the cases

$$(2) \qquad x'' + d(t, x, x')x' + g(t, x) = p(t)$$

and

$$(3) \qquad x'' + d(x, x')x' + g(x) = p(t).$$

For (3), the hypotheses are boundedness of d and p, periodicity of p with period T, the (crucial) growth condition that $x^{-1}g(x) \geqq a|x|^{\sigma}$ $(\sigma > 0)$ for large $|x|$, and symmetry conditions (e.g., $d(-x, v) = -d(x, v)$, $g(-x) = -g(x)$, $p(-t) = -p(t)$) which permit periodicity of a solution to be inferred from certain boundary conditions (such as $x(0) = x(T/2) = 0$ for the case cited). The existence of periodic solutions is deduced from the author's results on nonlinear second-order boundary-value problems [Math. Ann. **134** (1957), 167–194; MR **19**, 1054].

The author then shows that, with the additional hypotheses that T is the least period of f in (1), and that (1) has no periodic solutions whose least period is a non-integral multiple of T, there exist infinitely many solutions of (1) of least period kT for any integer $k \geqq 1$. The results are also stated for (2) and (3).

W. S. Loud (Minneapolis, Minn.)

2781:

Morimoto, Hiroshi. On the periodic solution of a certain non-linear differential equation. Math. Japon. **5** (1958/59), 39–43.

The author considers the differential equation

$$x'' + f(x, t)x' + g(x, t) = p(t)$$

under the following assumptions: (1) f, g are lipschitzian in x, continuously differentiable and periodic of period 2π in t, $g \operatorname{sgn} x \geqq b_0$ for $|x| > c_0$ for some positive b_0, c_0, and $p(t)$ is continuous and periodic of period 2π, $\int_0^{2\pi} p(t)dt = 0$; (2) $F(x, t) = \int_0^x f(x, t)dx$, $G(x, t) = \int_0^x g(x, t)dx$ satisfy $F \operatorname{sgn} x \to \infty$ with $|x|$, $G > 0$ for $x \neq 0$, and

$$F \operatorname{sgn} x \geqq b_1^{-1}|G_t + 2^{1/2}|F_t|\,|\cdot|g|^{-1}$$

for $|x| > c_1$ for some positive $b_1 < 1$, c_1. Using $V^2 = y^2 + 2G$ as gauge function for the solutions of the system

$$x' = y - F(x, t) + \int_0^t p(t)dt, \quad y' = F_t(x, t) - g(x, t)$$

and a standard fixed point argument, he proves the existence of at least one periodic solution of period 2π.

H. A. Antosiewicz (Los Angeles, Calif.)

2782:

Meisters, G. H. On almost periodic solutions of a class of differential equations. Proc. Amer. Math. Soc. **10** (1959), 113–119.

Let C^n denote an n-dimensional complex vector space with the usual norm and metric defined by this norm. Consider the vector differential equation $dx/dt = F(t, x)$, $-\infty < t < +\infty$, in which $F(t, x)$ satisfies a Lipschitz condition in x and is an almost periodic function in C^n for each x in some connected open set D of C^n. If $x(t)$ is a solution of this equation defined for all real t and if D contains the closure of the range of $x(t)$, then $x(t)$ is almost periodic if, and only if, $x(k)$, $k = 0, \pm 1, \pm 2, \cdots$, is an almost periodic sequence in C^n.

W. R. Utz (Columbia, Mo.)

2783:

Urabe, Minoru. A method of numerical calculation of non-linear vibration. Sûgaku **9** (1957/58), 201–218. (Japanese)

An expository paper on the periodicity of solutions of ordinary differential equations, summing up results of papers quoted in the references. Extensive details are given on a method of numerical estimates of periodic solutions of Van der Pol equations.

H. Yamabe (Minneapolis, Minn.)

2784:

Sispanov, Sergio. Free oscillations in the case of a resistance proportional to the square of the velocity. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 49–65. (Spanish)

The equation $mx'' = -kx \mp lx'^2$ can be transformed into

$$\frac{dv}{d\xi} \mp \frac{1}{2}v + \frac{c^2}{2}\cdot\frac{\xi}{v} = 0,$$

where

$$a = \frac{m}{2l} > 0, \quad c = \frac{1}{l}\left(\frac{km}{2}\right)^{1/2} > 0,$$

$x = a\xi$, $dx/dt = v$. The transformed equation has solution

$$(*) \qquad v = c(1 \mp \xi - e^{\mp t})^{1/2}.$$

The nature of the oscillations is discussed in terms of (*). [See E. Kamke, *Differentialgleichungen, Lösungenmethoden und Lösungen*, New York, 1948; page 551, equation 6.45.]

R. T. Herbst (Winston-Salem, N.C.)

2785:

Pavlyuk, I. A. A necessary and sufficient condition for non-oscillation of solutions of a linear differential equation of second order. Kiïv. Derž. Univ. Nauk Zap. **16** (1957), no. 2 = Kiev. Gos. Univ. Mat. Sb. **9** (1957), 111–113. (Russian)

A solution of a linear differential equation of second order is said to be non-oscillating on an interval if it does not have more than one zero there, and it is said to be non-oscillating on $[a, \infty)$ if it is non-oscillating on $[b, \infty)$ for some sufficiently large $b > a$. The author considers the equation (*) $y'' + q(x)y = 0$, where q is continuous on $[a, \infty)$. He shows that necessary and sufficient for any solution of (*) to be non-oscillating on $[c, \infty)$ $(c \geqq a)$ is the existence of a positive bounded function $\phi \in C^2$ on $[a, \infty)$ such that

$$q \leqq \frac{\phi''}{2\phi} - \frac{3}{4}\left(\frac{\phi'}{\phi}\right)^2$$

on $[a, \infty)$.

E. A. Coddington (Los Angeles, Calif.)

2786:

Volosov, V. M. Oscillation equations with slowly variable parameters. Dokl. Akad. Nauk SSSR **121** (1958), 22–25. (Russian)

Consider the system of equations

$$(*) \quad \frac{d}{dt}[m(\mu)\dot{x}] + Q(\mu, x) = \varepsilon f(x, \dot{x}, \mu), \quad \dot{\mu} = \varepsilon\phi(x, \dot{x}, \mu),$$

where ε is a small parameter, $\mu = (\mu_1, \cdots, \mu_n)$ and $\phi = (\phi_1, \cdots, \phi_n)$, subject to the initial conditions

$$(**) \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \quad [x_0^2 + \dot{x}_0^2 \neq 0], \quad \mu(0) = \mu_0.$$

It is assumed that $\operatorname{sign} Q = \operatorname{sign} x$ and that the equation $m(\mu)\ddot{x} + Q(\mu, x) = 0$ (μ an independent parameter) has only periodic solutions for the initial values given by (**). Then, if certain additional conditions are satisfied, the solution $x(t, \varepsilon)$ of (*), (**) will oscillate near the equilibrium position $x = 0$ for sufficiently small ε on a time interval

whose length is $O(1/\varepsilon)$. The author develops formulas for computing the amplitude of the oscillations and the range of values of μ to within an error of $O(\varepsilon)$. These results extend the author's earlier work [same Dokl. **106** (1956), 7–10; **114** (1957), 1149–1152; **115** (1957), 20–22; **117** (1957), 927–930; MR **18**, 41; **20** #1822; **19**, 276; **20** #1823]. *D. G. Aronson* (Minneapolis, Minn.)

2787:

Cronin, Jane. Note to Poincaré's perturbation method. Duke Math. J. **26** (1959), 251–262.

Consider the system of equations $dx/dt = F(x, t, \varepsilon)$ where x and F are n-vectors and F has period $T(\varepsilon)$ in t. Assuming that when $\varepsilon = 0$ the system has a solution of period $T(0)$, the problem is to find near-by solutions of period $T(\varepsilon)$ for the perturbed system. A necessary condition for the existence of such solutions is that the equations of first variation also have a solution of period $T(0)$, and if they have exactly one linearly independent solution of period $T(0)$ then this is also a sufficient condition. In the present paper the degenerate case (the problem of bifurcation) is attacked by first reducing the condition of periodicity to an amenable set of equations and then showing the existence of solutions by means of topological degree. The paper concludes with several examples. *G. Hufford* (Seattle, Wash.)

2788:

Yeh, Yen-chien. Periodic solutions and limit cycles of certain non-linear differential systems. Sci. Record (N.S.) **1** (1957), 391–394.

The author discusses qualitative problems on the existence, number, location, etc., of limit cycles of $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, P, Q being polynomials of the second degree. *J. L. Massera* (Montevideo)

2789:

Yeh, Yen-chien. Limit cycles of certain nonlinear differential systems. II. Sci. Record (N.S.) **2** (1958), 276–279.

Following his early work [see above review], the author considers one of the normal forms which can be obtained through affine transformations, namely $\dot{y} = x$, $\dot{x} = -y + dx - lx^2 + mxy + ny^2$. Theorems on the number of limit cycles which possibly exist, depending on the values of the coefficients, are established. *J. L. Massera* (Montevideo)

2790:

Chang, H. M. On the upward stability of maximal characteristic exponent and lower stability of minimal characteristic exponent. Acta Math. Sinica **8** (1958), 324–332. (Chinese. English summary)

Let $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ be the characteristic exponents of the linear homogeneous system $dx/dt = A(t)x$ of the nth order, $\Lambda = \max \lambda_i$, $\mu = \min \lambda_i$. Let λ be the characteristic exponent of any solution of the non-linear system $dx/dt = A(t)x + F(t, x)$. Λ [μ] is called upward stable [lower stable] if to each $\varepsilon > 0$ there is a $\delta > 0$ such that, when $\|F(t, x)\| \leq \delta \|x\|$ for all $t \geq 0$, $\lambda \leqq \Lambda + \varepsilon$ [$\lambda \geqq \mu - \varepsilon$]. Here $\|x\| = \sum_{i=1}^{n} |x_i|$. This paper extends the investigations carried out by B. F. Bylov [Dokl. Akad. Nauk SSSR **103** (1955), 181–184; MR **17**, 37]. Several sufficient conditions for upward and lower stability are given. For example: Let F be real and satisfy $\|F(t, x)\| \leq c\|x\|$ for all $t \geq 0$, where c is positive. Let $A = (a_{ij})$. Then Λ is upward stable and μ is lower stable if (i) $\lim a_{ik}(t) = 0$, $i \neq k$, and (ii) to each $\varepsilon > 0$, there is a $N > 0$ such that

$$\left| (t_2 - t_1)^{-1} \int_{t_1}^{t_2} a_{kk}(t)dt + \lambda_k \right| < \varepsilon$$

whenever $t_2 - t_1 \geq N$. *Choy-tak Taam* (Washington, D.C.)

2791:

★Malkin, Joel G. Theorie der Stabilität einer Bewegung. In deutscher Sprache herausgegeben von W. Hahn und R. Reissig. R. Oldenbourg, Munich, 1959. xiii + 402 pp. DM 47.00.

This is a translation of the (1952) Russian edition [*Teoriya ustoičivosti dviženiya*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; MR **15**, 873] to which have been appended an index and a bibliography. The latter includes a selected number of papers dated through 1956, which are referred to by brief footnotes in the text. Thus much of the progress that has been made in recent years is not indicated at all. Nevertheless this translation will undoubtedly contribute much to making Lyapunov's method more generally available. *H. A. Antosiewicz* (Los Angeles, Calif.)

2792:

Žen', Da-syun. Behavior of integral curves in the plane under Lyapunov stability, and necessary and sufficient conditions for Lyapunov stability and their application. Sci. Record (N.S.) **2** (1958), 285–288. (Russian)

Certain fairly obvious criteria for the stability and instability of a critical point of a second order autonomous system are established. *J. L. Massera* (Montevideo)

2793:

Lefschetz, Solomon. The ambiguous case in planar differential systems. Bol. Soc. Mat. Mexicana (2) **2** (1957), 63–74.

Barocio [*Contributions to the theory of nonlinear oscillations*, Vol. 3, pp. 127–135, Princeton Univ. Press, 1956; MR **19**, 145] has analyzed critical points of analytical differential systems (*) $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$, where X and Y are holomorphic at (0, 0) and vanish there. There is one ambiguous case, typified by a sector on the first quadrant bounded by two odd branches of $Y = 0$ such that $\dot{x} > 0$ everywhere in the sector, $\dot{y} < 0$ outside the sector and $\dot{y} > 0$ inside the sector. There are two possible configurations of solutions inside the sector, one including a "fan", and the other not. In the present paper the author shows how the problem of distinguishing between the two possibilities may always be resolved. The system (*) is transformed by a finite number of changes of variables of one of the two following types: $x = x_1^s$, $y = x_1^r y_1$ or $x = x_1^s$, $y = y_1 + \omega(x_1)$ with ω a power series. It is always possible to obtain a finite number of new systems for which the ambiguity can be resolved, and from this the original ambiguity can be resolved. The general technique is as in an earlier paper by the author [same Bol. (2) **1** (1956), 13–27; MR **18**, 481]. *W. S. Loud* (Minneapolis, Minn.)

2794:

★**Lefschetz, Solomon. On the critical points of a class of differential equations.** Contributions to the theory of nonlinear oscillations, Vol. IV, pp. 19–28. Annals of Mathematics Studies, no. 41. Princeton University Press, Princeton, N.J., 1958. ix+211 pp. $3.75.

The author considers the behaviour near the origin of the system

$$\frac{dx}{dt} = -y+C(x), \quad \frac{dy}{dt} = [y^2-2A(x)y+B(x)]E(x, y)$$

where A, B, C, E are convergent power series with A, B, C, B', C' vanishing at the origin, $E(0, 0)=1$. It is shown that the above system presents at most one sector of nested ovals; if such sector exists it must be crossed by the y axis. The main point of the proof is the fact that no sector of nested ovals can be on just one side of the y axis. The paper is closely related to a previous one of Barocio [reference in review above].

M. M. Peixoto (Baltimore, Md.)

2795:

Erugin, A. N. Structure of an explicit solution of the equation of the integral curves of a class of systems of two differential equations with non-holonomic right-hand sides. Vesci Akad. Navuk BSSR. Ser. Fiz.-Tèhn. Navuk **1958**, no. 4, 23–32. (Russian)

Explicit solution with a wealth of formulas of

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

($P(0, 0)=Q(0, 0)=0$) of the well-known center-focus situation when P, Q cease to be holomorphic at the origin [References: Erugin, Dokl. Akad. Nauk BSSR **2** (1958), no. 6, 234–237; ibid. no. 11]. *S. Lefschetz* (Mexico, D.F.)

2796:

Miščenko, E. F.; and Pontryagin, L. S. A proof of some asymptotic formulas for solutions of differential equations involving a small parameter. Dokl. Akad. Nauk SSSR **120** (1958), 967–969. (Russian)

The second author [Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 605–626; MR **20** #1029a] computed a formal asymptotic expansion of the solution of

$$\varepsilon\dot{x}^i = f^i(x^1, \cdots, x^k, y^1, \cdots, y^l),$$
$$\dot{y}^j = g^j(x^1, \cdots, x^k, y^1, \cdots, y^l)$$
$$(i = 1, \cdots, k; j = 1, \cdots, l),$$

where ε is a small parameter, in the neighborhood of a point at which $\det \|\partial f^i/\partial x^\alpha\| = 0$. This expansion was used both in the paper cited above and in a paper by the first author [ibid. **21** (1957), 627–654; MR **20** #1029b]. In this note the authors sketch a proof that the expansion in question does indeed approximate the true solution with the required accuracy.

D. G. Aronson (Minneapolis, Minn.)

2797:

El'sgol'c, L. È. Approximate methods for integration of differential-difference equations. Advancement in Math. **4** (1958), 188–197. (Chinese)

This is a Chinese translation of the author's paper that appeared originally in Uspehi Mat. Nauk (N.S.) **8** (1953), no. 4(56), 81–93 [MR **15**, 629].

Choy-tak Taam (Washington, D.C.)

2798:

El'sgol'c, L. È. Stability of solutions of differential-difference equations. Advancement in Math. **3** (1957), 562–576. (Chinese)

This is a Chinese translation of the author's expository paper that appeared originally in Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4 (62), 95–112 [MR **17**, 44].

Choy-tak Taam (Washington, D.C.)

2799:

Chin, Yuan-shun. On the equivalence problem of differential equations and difference-differential equations in the theory of stability. Acta Math. Sinica **8** (1958), 457–472. (Chinese. English summary)

The author shows that if the time-lag d is sufficiently small, then the solution of $u'(t)=pu(t)+qu(t-d)$ is stable, provided that $p+q<0$. The result is then applied to a nonlinear stability problem.

R. Bellman (Santa Monica, Calif.)

2800:

Hille, Einar. Remarques sur les systèmes des équations différentielles linéaires à une infinité d'inconnues. J. Math. Pures Appl. (9) **37** (1958), 375–383.

Pour attirer l'attention sur la différence essentielle qui existe entre les équations différentielles à un nombre fini d'inconnues et celles à un nombre infini d'inconnues, l'A. considère, dans l'espace (l) des séries absolument convergentes, les deux systèmes (A) $z_k'(t)+kz_k(t)=\sum_{m=k+1}^{\infty}z_m(t)$ $(k=1, 2, \cdots)$, (B) $z_k'(t)+k^2z_k(t)=\sum_{m=k+1}^{\infty}z_m(t)$ $(k=1, 2, \cdots)$, où t est un nombre complexe, et met en évidence certains faits singuliers concernant les solutions de ces systèmes. En voici deux: 1° Il existe des solutions nulles $\{z_k(t)\}$ de (A) dont les $z_k(t)$ sont entiers (en t), tandis que le vecteur $z_k(t) \in (l)$ ($\mathrm{Re}\, t>0$) admet l'axe imaginaire comme ligne singulière. 2° Il existe une infinité non dénombrable de solutions $\in (l)$ linéairement indépendantes, fonctions entières en $(t-t_0)^{-1}$ quel que soit t_0 réel.

C. Foiaş (Bucharest)

2801:

Schäffer, J. J. On linear differential equations with constant coefficients in Banach spaces. Bol. Fac. Ingen. Agrimens. Montevideo **6** (1957/58), 351–356. Also published as Fac. Ingen. Montevideo. Publ. Inst. Mat. Estadist. **3** (1958), 105–110. (Spanish. English summary)

The author considers a differential equation $\dot{x}+Ax=f(t)$, where $0 \leq t < \infty$, x and $f(t)$ are in a complex Banach space X, and A is a linear operator of X into itself. Write X_0 for the set of values taken on at $t=0$ by the bounded solutions of the homogeneous equation ($f(t)=0$). Theorem: The necessary and sufficient condition that X_0 be a closed manifold and have a closed complement, and that the above equation have at least one bounded solution $x(t)$ for each bounded and continuous $f(t)$, is that the spectrum of A should not intersect the imaginary axis. The result also holds for $f \in L^p$, $1<p\leq\infty$, and other cases. The real case is treated by immersing X into a complex space in the standard way. The proof uses standard constructions of the operational calculus and also some previous theorems of J. L. Massera and the author [Ann. of Math. (2) **67** (1958), 517–573; MR **20** #3466].

E. R. Lorch (New York, N.Y.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 2763, 2769, 2849, 3015.

2802:

★**Bateman, H. Partial differential equations of mathematical physics.** Cambridge University Press, New York, 1959. xii+522 pp. Paperbound: $4.95.

An unchanged reprint of the first edition [University Press, Cambridge, 1932].

2803:

Diaz, J. B. On an analogue of the Euler-Cauchy polygon method for the numerical solution of $u_{xy}=f(x, y, u, u_x, u_y)$. Arch. Rational Mech. Anal. **1** (1958), 357–390.

"This paper develops, with an eye on the numerical applications, an analogue of the classical Euler-Cauchy polygon method (which is used in the solution of the ordinary differential equation $dy/dx=f(x, y)$, $y(x_0)=y_0$) for the solution of the following characteristic boundary value problem for a hyperbolic partial differential equation:

$$u_{xy} = f(x, y, u, u_x, u_y),$$
$$u(x, y_0) = \sigma(x),\ u(x_0, y) = \tau(y),$$

where $\sigma(x_0)=\tau(y_0)$. The method presented here, which may be roughly described as a process of bilinear interpolation, has the advantage over previously proposed methods that only the tabulated values of the given functions $\sigma(x)$ and $\tau(y)$ are required for its numerical application. Particular attention is devoted to the proof that a certain sequence of approximating functions, constructed in a specified way, actually converges to a solution of the boundary value problem under consideration. Known existence theorems are thus proved by a process which can actually be employed in numerical computation." (From the author's abstract.)

E. A. Coddington (Los Angeles, Calif.)

2804:

Rhoades, B. E. On forming partial differential equations. Amer. Math. Monthly **66** (1959), 473–476.

Partial differential equations are obtained by differentiation for functions z represented by

$$z(x, y) = \sum_{i=1}^{n} u_i(x, y)\ F_i(v_i(x, y))$$

and similar expressions, where $u_i(x, y)$ and $v_i(x, y)$ are known functions and the F_i are arbitrary, all of them possessing the right number of derivatives. For $n=2$ a necessary and sufficient condition is given for the existence of a second order partial differential equation satisfied by z.

M. Steinberg (Los Angeles, Calif.)

2805:

Levitan, B. M. Certain questions in the spectral theory of self-adjoint differential operators. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 6 (72), 117–144. (Russian)

This is a survey of results in the theory of eigenfunction expansions obtained since 1950, with special emphasis upon topics in which the author has worked. The discussion is divided into 6 chapters. Chapter 1 gives a brief statement of the results in eigenfunction expansions for ordinary self-adjoint differential operators obtained from the spectral theorem for unbounded self-adjoint operators in Hilbert space by M. Kreĭn's method of directed functionals [Dokl. Akad. Nauk SSSR **53** (1946), 3–6; MR **8**, 277]. Chapter 2, which is considerably longer, is concerned with various 'direct' proofs of theorems in the same sphere, i.e. proofs not using the abstract theory of operators in Hilbert space, given by Titchmarsh [*Eigenfunction expansions associated with second-order differential equations*, Clarendon Press, Oxford, 1946; MR **8**, 458], the author [*Razloženie po sobstvennym funkciyam differencial'nyh uravneniĭ vtorogo poryadka*, Gosudarstv. Izdat. Tehn.-Teor Lit., Moscow-Leningrad, 1950; MR **12**, 183], N. Levinson [Duke Math. J. **18** (1951), 57–71; MR **12**, 828] and K. Yosida [Nagoya Math. J. **1** (1950), 49–50; MR **13**, 39]. {Reviewer's remark: The author somewhat slights the abstract approach of Kodaira [Amer. J. Math. **72** (1950), 502–544; MR **12**, 103], which seems in connection with Chap. 1 to be independent of Kreĭn's announcement, and was the only complete and explicit publication of a proof of these results at the time.} Chapter 2 concludes with a brief description of various results on the Green's function, essential self-adjointness, and asymptotic behavior of eigenvalues in the case of self-adjoint ordinary differential operators.

Chapter 3 begins the consideration of partial differential operators and opens with a summary of Carleman's paper on the Schroedinger equation [Ark. Mat. Astr. Fys. **24** (1934), no. 11]. It concludes with a brief listing of results extending the theory of eigenfunction expansions to self-adjoint differential operators, for the Schroedinger equation by Titchmarsh [loc. cit.] and Povzner [Mat. Sb. (N.S.) **32** (74) (1953), 109–156; MR **14**, 755], to general elliptic operators by L. Gårding [Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 44–45, Lunds Univ. Mat. Inst., 1954; MR **17**, 158] and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 454–458, 459–463; MR **16**, 134] based upon an earlier abstract argument of F. Mautner [ibid. **39** (1953), 49–53; MR **14**, 659] and its extension to general differential operators by Gel'fand and Kostyučenko [Dokl. Akad. Nauk SSSR **103** (1955), 349–352; MR **17**, 388]. Chapter 4, which is somewhat longer, is concerned with the behaviour of the spectral function $\theta(x, y, \lambda)=\sum_{\lambda_n<\lambda}\Phi_n(x)\Phi_n(y)$, where the Φ_n are the eigenfunctions of the Laplace or Schroedinger operators on a bounded domain, and outlines the author's method of investigating the asymptotic behaviour of this function for large λ by the use of related properties of a corresponding wave equation. Chapter 5 deals with results obtained for the asymptotic behaviour of the eigenvalues of the Schroedinger operator on an unbounded domain under Dirichlet boundary conditions with a potential which makes the spectrum discrete. Chapter 6 deals with the results obtained for the inverse Sturm-Liouville problem for second-order ordinary differential operators, and in particular with the methods of the paper of Gel'fand and Levitan [Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 309–360; MR **13**, 558], as well as related results of M. Kreĭn [Dokl. Akad. Nauk SSSR **76** (1951), 345–348; **82** (1952), 669–672; MR **13**, 43; **14**, 649]. {Reviewer's remark: It is rather surprising that the author does not mention the extensions of these results to the Schroedinger operator given by Berezanskiĭ [ibid. **93** (1953), 591–594; **105** (1955), 197–200; MR **15**, 797; **17**, 1210].}

F. Browder (New Haven, Conn.)

2806:

Levitan, B. M. Differentiation of an expansion in eigenfunctions of the Schrödinger equation. Trudy Moskov. Mat. Obšč. **7** (1958), 269–290. (Russian)

Let D be a finite domain in the three-dimensional real space R_3, having B as its boundary. The author considers the problem $-\Delta u+q(x)u=\lambda u$, $\partial u/\partial n=0$ on B. The spectrum is assumed non-negative, with eigenfunctions ϕ_n and corresponding eigenvalues $\lambda_n=\mu_n^2$. Associated with any $f\in L^2(D)$ is its eigenfunction expansion $\sum_{n=1}^{\infty}(f,\phi_n)\phi_n$. The main results proved concern the Riesz summability of the differentiated series to the derivative of f. Indeed if $f\in L^2(D)$, and in a neighborhood of a point x has continuous partial derivatives up to order α inclusive, and q has bounded partial derivatives up to order α inclusive in a vicinity of x, then

$$\lim_{\mu\to\infty}\sum_{\mu_n<\mu}\left(1-\frac{\mu_n^2}{\mu^2}\right)^{\alpha+1}c_nD^\alpha\phi_n(x)=D^\alpha f(x).$$

Here D^α is any partial derivative of order α, $c_n=(f,\phi_n)$.

The proof uses a number of estimates, and a Tauberian theorem previously used by the author in connection with the Laplace operator. The case when $q(x)\to\infty$, $|x|\to\infty$, and D is all of R_3, is also considered. The result is somewhat similar.

E. A. Coddington (Los Angeles, Calif.)

2807:

Titchmarsh, E. C. On the eigenvalues in problems with spherical symmetry. II. Proc. Roy. Soc. London Ser. A **251** (1959), 46–54.

[For part I, see same Proc. **245** (1958), 147–155; MR **20** #2530.] This paper is concerned with the eigenvalues of

$$\frac{d^2\psi}{dr^2}+\left\{\lambda-q(r)-\frac{l^2+l}{r^2}\right\}\psi=0\quad(0<r<\infty),$$

where λ is large and l varies over a certain large range $0\leq l\leq L$. It is assumed that $q(r)$ is three times differentiable, that $q(r)$ and $q'(r)$ are non-decreasing, and that $q'(r)/q(r)$, $q''(r)/q'(r)$ and $q'''(r)/q''(r)$ are all $O(1/r)$ as $r\to\infty$. It is shown that the eigenvalues $\lambda_{l,n}$ satisfy equations of the form

$$\int_{R'}^{R}\left\{\lambda_{l,n}-q(r)-\frac{l^2+l}{r^2}\right\}^{1/2}dr=(n+\tfrac{1}{2})\pi+O\left\{\frac{L}{l(L-l)}\right\},$$

where R', R are the zeros of $\lambda-q(r)-(l^2+l)/r^2$.

E. T. Copson (St. Andrews)

2808:

Rasulov, M. L. A formula for the expansion of an arbitrary function in a series of eigenfunctions of a certain class of boundary problems with a parameter for linear partial differential equations. Dokl. Akad. Nauk SSSR **120** (1958), 252–255. (Russian)

The author extends the result of Tamarkin [Math. Z. **27** (1928), 1–54] for the expansion of a function in L^2 in terms of solution functions to ordinary linear differential equations to the case of partial differential equations with mixed type boundary conditions. The differential equation involves two differential operators whose domains are disjoint but whose Cartesian product makes up the entire space. The expansion is obtained through use of the residues of the two Green's functions.

R. G. Langebartel (Urbana, Ill.)

2809:

Payne, L. E.; and Weinberger, H. F. New bounds for solutions of second order elliptic partial differential equations. Pacific J. Math. **8** (1958), 551–573.

Il presente lavoro reca interessanti contributi al problema consistente nell'assegnare limitazioni integrali "a priori" per le soluzioni dei problemi al contorno relativi alle equazioni differenziali lineari. Viene considerato il caso di una equazione di secondo ordine di tipo ellittico autoaggiunta e di sistemi di equazioni di tipo ellittico del secondo ordine verificanti una particolare condizione di simmetria. Le formole di maggiorazione vengono ottenute usando una identità integrale di per sè notevole e che estende una precedentemente data da Rellich [Math. Z. **46** (1940), 635–646; MR **2**, 56] per l'operatore di Laplace. I problemi al contorno presi in esame sono quello interno di Dirichlet ed il problema esterno di tipo misto Dirichlet-Neumann. A titolo di esempio mostriamo una delle più semplici maggiorazioni ottenuta dagli AA., relativa al problema $u=0$ su B, $\mathscr{A}(u)=f$ in D (D dominio regolare limitato di frontiera B, $\mathscr{A}$ operatore ellittico autoaggiunto del secondo ordine). Essa è

$$(*)\qquad \int_B\left(\frac{\partial u}{\partial \nu}\right)^2dS\leq K\iint_D f^2dV;$$

essendo K una costante che si calcola esplicitamente in funzione di un vettore avente componente normale positiva su B ($\partial/\partial\nu$ derivazione conormale su B). Dalle maggiorazioni globali, usando una parametrice di E. E. Levi, gli AA. deducono maggiorazioni puntuali. È da notare che diverse delle formole di maggiorazione ottenute dagli AA. sono contenute o sono analoghe (ad esempio la (*)) ad altre stabilite in precedenza, con procedimenti non dissimili, dal recensore [cfr. Ann. Mat. Pura Appl. (4) **36** (1954), 273–296; *Proc. Internat. Congr. Mathematicians, Amsterdam, 1954*, Vol. 3, pp. 216–228, Noordhoff, Groningen, 1955; Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 411–418; *Atti Convegno Internat. Equazioni Lineari alle Derivate Parziali, Trieste, 1954*, pp. 174–227, Edizioni Cremonese, Rome, 1955; MR **16**, 374; **19**, 969; **18**, 131; **17**, 626]. Viene anche mostrato come le maggiorazioni ottenute consentono di limitare l'errore di approssimazione nel metodo di Rayleigh-Ritz (dei minimi quadrati). La questione della convergenza del metodo (dimostrazione dell'esistenza di particolari sistemi completi di funzioni) non è però considerata.

G. Fichera (Madison, Wis.)

2810:

Lewy, Hans. On the reflection laws of second order differential equations in two independent variables. Bull. Amer. Math. Soc. **65** (1959), 37–58.

The author studies the analytic continuation of solutions of elliptic differential equations in two independent variables across an analytic boundary on which they satisfy further analytic equations connecting the value of the solution and those of its first partial derivatives. A typical and important result is the theorem: "Let a, b, and c be real analytic functions of x and y in a circle γ around the origin; let $u(x, y)$ satisfy the equation $\nabla^2u+au_x+bu_y+cu=0$ in the part $y<0$ of γ and be continuous with first derivatives in $y\leq 0$. We suppose that u satisfies

on $y=0$ a real relation $u_y=h(x, u, u_x)$ where h is an analytic function of its arguments near $x=0$, $u=u(0, 0)$, $u_x=u_x(0, 0)$. Then, u can be continued analytically across $y=0$ near the origin." This theorem plays a role in free boundary problems of fluid dynamics. The method of proof is a passage through the complex and use of the Riemann function with complex argument. The extension of the solution is carried out by means of a functional equation and by using the concept of terminal introduced by the author. It is defined as follows: Let $\Omega[\phi(t)|(p), z]$ be a functional defined for all continuous complex-valued functions $\phi(z)$ in a simply-connected domain D and for all rectifiable curves (p) in D leading from the origin $0 \in D$ to the point $z \in D$. Ω is called a terminal if, whenever $\phi(z)$ is analytic in D, the value $\chi(z)=\Omega[\phi(t)|(p), z]$ is independent of (p). A terminal is called regular if $\chi(z)$ is a regular function of z whenever $\phi(z)$ is regular in D. The functional equation for $u(x, y)$ may then be reduced to a terminal equation $\phi(z)=K(z)+\Omega[\phi(t)|(p), z]$ with given $K(z)$. Simple conditions are given in order that the solution of this implicit equation for $\phi(z)$ be a regular terminal of $K(z)$. The concept of terminal may obviously be applied in many analogous continuation problems, even in the non-linear case. Finally, a fundamental difference between problems involving two or three independent variables is pointed out. A function $U(x, y, z)$ is constructed which is harmonic in a half-space bounded by a plane p, which is continuous up to and including p (except for one point 0) and satisfies on $p-0$ a linear homogeneous boundary condition of first order with constant coefficients. But $U(x, y, z)$ cannot be continued across p as a harmonic function into a domain which contains a line through 0 of some fixed but arbitrary small angle with p. This situation is entirely different from the two-dimensional case when under similar data we can extend harmonic functions by reflection in the large. *M. Schiffer* (Stanford, Calif.)

2811:

Hirasawa, Yoshikazu. On the bounded solutions of the partial differential equation $\Delta u=f(x, y, u, p, q)$. Comment. Math. Univ. St. Paul. **7** (1959), 13–19.

The author studies bounded, regular (meaning continuous first derivatives) solutions of the differential equation appearing in the title, where f is a continuous function of its arguments, bounded when u is bounded, and (x, y) lies in a bounded domain. The Laplacian Δu is taken in a generalized sense as the limit (when it exists) either of the usual five-point difference formula or of the appropriate difference quotient of the function and its circular mean. [See T. Satô, Compositio Math. **12** (1954), 157–177; MR **17**, 474; and S. Simoda, Osaka Math. J. **6** (1954), 243–268; MR **16**, 706.] The paper is devoted mainly to proving comparison theorems between regular solutions u of the differential equation and other regular functions ω satisfying the differential inequality $\Delta\omega<f(x, y, u, p, q)$ when $\omega<u$ (or more generally $\Delta\omega<f$). A typical result is that if in addition $\omega-u$ has only non-negative limiting values on the boundary, with the possible exceptions of certain sets of zero capacity, then $\omega \geqq u$ in D. There is also a study of the problem in the circle, and some theorems of Fatou type concerning the boundary values of u are given. *J. W. Green* (Los Angeles, Calif.)

2812:

Zaidman, S. Sur la presque-périodicité des solutions de l'équation des ondes non homogène. J. Math. Mech. **8** (1959), 369–382.

Consider the non-homogeneous partial differential equations

$$u_{tt}(X, t)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(X)\frac{\partial u}{\partial x_j}\right)-a(X)u(X, t)+f(X, t),$$

$$a(X)\geqq 0,$$

for X in a domain Ω, and boundary condition $u|_S=0$; $f(X, t)$ being C^1 in the closure of $\Omega\times(-T, T)$ for every $T>0$. Now, if $f(X, t)$ is almost periodic in t as a function with values in $C^0(\bar{\Omega})$; if $u(X, t)$ is a "classical" solution of the boundary value problem (the paper defines "classical"); and if in the "energy" metric

$$\|u(\cdot, t)\|_E^2=\int_\Omega\left\{\sum_{i,j=1}^{n}a_{ij}(X)\frac{\partial u}{\partial x_i}\cdot\frac{\partial\bar{u}}{\partial x_j}+a(X)|u|^2+|u_t|^2\right\}dX,$$

the range of $u(X, t)$ as function of t is relatively compact; then $u(X, t)$ is almost periodic in t. {Reviewer's remark: The paper is more or less a later version of previous ones by the author. The assumption $a(X)\geqq 0$ makes the result a very non-obvious one.} *S. Bochner* (Princeton, N.J.)

2813:

Yamabe, Hidehiko. Kernel functions of diffusion equations. I. Osaka Math. J. **9** (1957), 201–214.

This paper presents a new, elegant method for constructing Green's function G for the heat equation over any domain D in Euclidean space. $G(x, y)$ is obtained as $\lim (E_{t/n}{}^*)^n$, where $E_t(x, y)$ is Green's function for the heat equation over the whole space, and the $*$ operation is a kind of convolution, where integration is carried out only over the domain D. The author shows that G vanishes in all boundary points of D which can be reached by interior spheres.

P. D. Lax (New York, N.Y.)

2814:

Višik, M. I.; and Lyusternik, L. A. Stabilization of solutions of parabolic equations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 273–275. (Russian)

Applying the results obtained in an earlier note on operator equations [same Dokl. **111** (1956), 12–15; MR **19**, 861], the authors consider the second-order parabolic equation $Lu=\partial u/\partial t+A(t)u=f(x, t)$, with $A(t)u=-a(t)\Delta u+\sum_j b_j(x, t)D_ju+c(x, t)u$, and give conditions on the coefficients $a(t)$, b_j, and c so that a solution $u(x, t)$ of the equation $Lu=f(x, t)$ on a cylindrical domain $G\times R^+$ satisfying the condition $u=0$ on the sides of the cylinder, and the corresponding solution $v(x, t)$ of the equation $A(t)u=f(x, t)$ in G under null Dirichlet boundary conditions (and t acting only as a parameter), should approach one another as $t\to\infty$, either in $L^2(G)$ or uniformly on G. The authors remark that the method could be applied to more general parabolic equations, but the relative complexity of the present results would seem to indicate that writing out comparable conditions for higher order equations in general might be quite a chore.

F. Browder (New Haven, Conn.)

2815:

Samarskiĭ, A. A. Parabolic equations with discontinuity coefficients. Dokl. Akad. Nauk SSSR **121** (1958), 225–228. (Russian)

Let $x=\eta_i(t)$ $(i=0, \cdots, n+1)$ on $0\leq t\leq T$ be a set of plane curves such that η_i' is Hölder continuous and $\eta_0(t)<\eta_1(t)<\cdots<\eta_{n+1}(t)$. Let

$$\Delta_i=\{\eta_i(t)<x<\eta_{i+1}(t),\ 0<t\leq T\};\quad D=\sum_{i=0}^{n}\Delta_i.$$

Consider the parabolic equation

$$(1)\qquad Lu=u_{xx}-u_t-a(x,t)u_x-b(x,t)u=-f(x,t),$$

where f, f_x, a, a_x, b, b_x are appropriately Hölder continuous in each Δ_i, subject to the conditions

$$(2)\qquad u(x,0)=\varphi(x),\quad u(\eta_0(t),t)=u_1(t),\quad u(\eta_{n+1}(t),t)=u_2(t),$$

$$(3)\qquad [u]_i=0,\quad [qu_x-ru]_i=0\quad (x=\eta_i(t);\ i=1,\cdots,n),$$

where φ, u_1, u_2, r, q are given functions satisfying certain regularity conditions in each Δ_i and $[u]_i=u(\eta_i+0,t)-u(\eta_i-0,t)$. $u(x,t)$ is said to be a regular solution of (1) if it satisfies (1) in D, is Hölder continuous in $\bar{D}$, and if u_x, u_{xx}, u_t satisfy Hölder conditions in each Δ_i. Under these and certain additional conditions on the data the author sketches a proof of the existence and uniqueness of a regular solution of (1), (2), (3). The method of proof is briefly as follows. Starting from the fundamental solution of the heat equation, the author constructs a singular solution $G(x,t,\xi,\tau)$ of the heat equation which satisfies certain compatibility conditions analogous to (3). With the aid of G he then obtains an integral equation for the solution of (1), (2), (3) which can be solved by successive approximations. *D. G. Aronson* (Minneapolis, Minn.)

2816:

★Garnir, H. G. Les problèmes aux limites de la physique mathématique. Introduction à leur étude générale. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Mathematische Reihe, Bd. 23. Birkhäuser Verlag, Basel-Stuttgart, 1958. 234 pp. 29 francs suisses.

In recent years the use of such tools as operators in Hilbert and Banach spaces, the theory of distributions and other methods of functional analysis has become commonplace in investigations of problems in partial differential equations. Despite the very wide literature in this field there are but few books dealing with the subject from this point of view. The present book is designed to help fill this gap. Its chief objective is the study of boundary value problems for equations of the form

$$\Box_{a,b,c}u\equiv a\frac{\partial^2u}{\partial t^2}+b\frac{\partial u}{\partial t}+cu-\Delta u=m(x,t),$$

where a, b and c are real constants, $a>0$ or $a=0$ and $b\geq 0$ and Δ is the Laplacian. Since the wave and heat equation are special cases of these equations as indeed are also the Laplace, Poisson and meta-harmonic equation, the author feels justified in using the title "Boundary value problems of mathematical physics". It is written for a reader who is presumed to have some familiarity with Lebesgue integration, functions of a complex variable, and Laplace and Fourier transforms.

The book is divided into four parts called books. Book I contains an introduction to the theory of Hilbert spaces which, despite its brevity (45 pages), starts with the axioms and manages to cover such topics as weak and strong derivatives of operators and of their inverses with respect to parameters, resolvents, etc. Book II is concerned with the Dirichlet-Neumann problem for the meta-harmonic operator $-\Delta+z$ where Δ is the Laplacian. An elementary (fundamental) solution for the meta-harmonic operator is derived here in a rather artificial way. The concept of weak solution is introduced. Using the fundamental solution and an auxiliary function constant in a neighborhood of a point, a representation for interior values of weak solutions is obtained. From this representation local differentiability and other estimates can be obtained. The Green's function for the Dirichlet-Neumann problem is treated.

Book III studies the Laplace transform and its inversion, the latter being defined as a distribution. Books I, II and III may be regarded as technical preparations for book IV. In the latter, solutions in the sense of a kind of distribution are defined for the equation $\Box_{a,b,c}u=m(x,t)$. Here a distribution is a mapping u_φ of the test functions $\varphi(t)$ into $L_2(\Omega)$. If there exists $u(x,t)$ so that $u_\varphi=\int u(x,t)\varphi(t)dt$, the solution is then called a "physical solution", otherwise only a "mathematical" one. A distribution solution must satisfy

$$u_{a(\partial^2\varphi/\partial t^2)-b(\partial\varphi/\partial t)+c\varphi}-\underline{\Delta}u_\varphi=\int_0^\infty m(x,t)\varphi(t)dt+(au_1+bu_0)\varphi(0)-au_{\varphi t}(0),$$

where $\underline{\Delta}u_\varphi$ denotes the weak Laplacian of u_φ. The author shows that distribution solutions always exist and are unique for properly formulated boundary problems. He continues with an investigation of the Green's function, local problems, properties and existence of physical solutions, etc.

There are a few misprints, and the theorem on p. 94 §38 seems wrong. However it does not play an essential role in what follows. On the whole, the exposition is clear, but the pace is rapid. It seems likely that a reader with only limited experience will have difficulty unless the book is used in connection with a course. Such basic subjects as the Cauchy-Kowalewski theorem, eigenfunctions, maximum principle, to name a few, are not mentioned.

As a means of introducing the reader to some of the modern methods, Prof. Garnir's book makes a valuable contribution to the literature.

A. N. Milgram (Berkeley, Calif.)

2817:

Garnir, H. G.; et Gobert, J. Le problème de Dirichlet-Neumann pour les opérateurs métaharmonique des ondes et de la diffusion par la méthode des fonctions propres. III. Bull. Soc. Roy. Sci. Liège **27** (1958), 119–127.

Questo lavoro fa parte di un gruppo di tre Note dallo stesso titolo, due delle quali sono già state recensite sulla MR [lo stesso Bull. **26** (1957), 279–289; **27** (1958), 17–27; MR **20** #1066]; a questa precedente recensione rinviamo per le notazioni e l'impostazione dei problemi trattati. In

questa III Nota viene approfondito lo studio del problema di Dirichlet-Neumann per l'equazione della diffusione

$$(1)\qquad \left(-\Delta+b\frac{\partial}{\partial t}+c\right)u=m \quad (b>0);$$

precisamente si dimostra anzitutto che la soluzione, precedentemente trovata nell'ambito delle distribuzioni, prende sempre la forma, detta dagli A. "fisica", $u[\varphi]=\int_0^\infty u(x,t)\varphi(t)dt$, dove $u(x,t)=bT_t u_0+T_t*m$, gli operatori $T_t u_0$ e T_t*m essendo definiti nel modo seguente: 1°) per ogni $u_0(x)\in L^2(\Omega)$

$$(2)\qquad T_t u_0=\frac{1}{b}\sum_{k=1}^{\infty}u_k(u_0,u_k)\exp\left(\frac{z_k-c}{b}t\right);$$

2°) per ogni $m(x,t)$ soddisfacente alle ipotesi introdotte nella Nota II per il secondo membro di (1)

$$(3)\qquad T_t*m=\frac{1}{b}\sum_{k=1}^{\infty}u_k\int_0^t(m(x,\tau),u_k(x))\exp\left(\frac{z_k-c}{b}(t-\tau)\right)d\tau.$$

Interessante è anche la seconda parte di questa Note, nella quale è data una dimostrazione diretta, senza ricorrere alla teoria generale svolta nelle Note I e II nel l'ambito delle distribuzioni, del fatto che $u(x,t)=bT_t u_0$ risolve il problema di Dirichlet-Neumann per l'equazione (1) omogenea ($m=0$), più precisamente nel senso che $u(x,t)$ è una funzione appartenente allo spazio N (condizione di Dirichlet-Neumann), derivabile rispetto a t in $L^2(\Omega)$, soluzione delle (1) con $m=0$ e tale che $u(x,t)\to u_0$ in $L^2(\Omega)$ per $t\to 0$. Questa possibilità di uno studio diretto della soluzione "fisica" non esiste però in generale quando è $m=0$, se non si introducono ipotesi assai restrittive sui dati, per la presenza dell'operatore T_t*m; e ciò conferma l'interesse delle trattazione delle Note I e II. Per uno studio dei problemi trattati in questa Nota e nelle precedenti, fatto in ipotesi più generali sul dominio Ω attraverso l'uso della trasformazione di Laplace nel campo delle distribuzioni, si veda la recente bella monografia di uno degli A. [#2816]. *E. Magenes* (Genoa)

2818:

Garnir, H. G. Problème de Dirichlet-Neumann pour l'opérateur de la diffusion. Bull. Soc. Math. Belg. **9** (1957), 31–41.

The diffusion equation on an open set in n-space, with prescribed initial values and Dirichlet-Neumann boundary conditions independent of time, is solved by means of the resolvent of the Laplacian operator with the same boundary conditions. The method uses results from the author's book [#2816] but is quite different from the treatment of the diffusion equation there. *G. A. Hunt* (Ithaca, N.Y.)

2819:

Browder, Felix E. A remark on the Dirichlet problem for non-elliptic self-adjoint partial differential operators. Rend. Circ. Mat. Palermo (2) **6** (1957), 249–253.

I. S. Louhivaara [same Rend. **5** (1956), 260–274; MR **19**, 281] has given a necessary and a sufficient condition for the solvability of the Dirichlet problem for a self-adjoint, not necessarily elliptic, second order differential equation.

The present paper gives a necessary and sufficient condition for the solvability of the Dirichlet problem for a self-adjoint operator of arbitrary even order. This condition implies the results of Louhivaara.

The operator A is defined by

$$Au=\sum_{|\alpha|,|\beta|\le m}D^\alpha a_{\alpha\beta}(x)D^\beta u,\quad a_{\alpha\beta}=a_{\beta\alpha},$$

on an open set G of E_n. The space $W^{m,2}(G)$ is the Hilbert space of functions with square integrable derivatives up to order m and norm

$$\|u\|^2=[u,u]=\sum_{|\alpha|\le m}(D^\alpha u,D^\alpha u),\quad (u,u)=\int_G|u|^2.$$

H is the subspace of $W^{m,2}$ which is the closure of the set C_c^∞ of infinitely differentiable functions vanishing outside a compact subset of G. If $a(u,v)=\sum(a_{\alpha\beta}D^\alpha u,D^\beta v)$, the Dirichlet problem $Au=f$ with $u=g\in W^{m,2}(G)$ on ∂G is reformulated as $v\in H$, $a(v,\Phi)=(f,\Phi)-a(g,\Phi)$ for all $\Phi\in C_c^\infty$. Then $u=v+g$. The right-hand side is a bounded linear functional in Φ. Hence there is a w depending on f,g such that $(f,\Phi)-a(g,\Phi)=[w,\Phi]$. The left-hand side is a bounded bilinear functional. Hence, there is a bounded self-adjoint transformation T in H such that $a(v,\Phi)=[Tv,\Phi]$. Thus, the Dirichlet problem is reduced to the equation $Tv=w$.

The author's result is: if the family of spectral projections of T is E_λ, then w is in the range of T if and only if $\int\lambda^{-2}d[E_\lambda w,w]<\infty$.

H. F. Weinberger (College Park, Md.)

2820:

Vishik, M. I.; and Lusternik, L. A. On some even order elliptic equations containing small parameters in higher derivative terms and degenerating into equations of order one or, generally, of an odd order. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 962–965. (Russian)

This is the second of a series of notes in which the authors consider the theory of singular perturbations of an elliptic operator with a small parameter multiplying the top-order terms. In their first note [Dokl. Akad. Nauk SSSR **113** (1957), 734–737; MR **19**, 861], they considered the Dirichlet problem for operators of the form $\varepsilon L+L'$, where L is elliptic of order $2m$ and L' is elliptic of lower order. In the present note, they discuss results in the case in which L' is no longer elliptic, and in particular the case in which L is order two and L' is of order one, obtaining in the latter case results extending those originally obtained by N. Levinson [Ann. of Math. (2) **51** (1950), 428–445; MR **11**, 439]. In the second section of the note, they indicate results for equations of the form $L_\varepsilon u_\varepsilon=\sum_{s=2k+1}^{2(k+j)}\varepsilon^{s-(2k+1)}L_s u_\varepsilon=h$, where for even s, L_s is elliptic of order s, while for odd s, L_s is of the form $\partial_1 M+M'$, M and M' elliptic of order $(s-1)$.

F. Browder (New Haven, Conn.)

2821:

Krylov, A. L. The first boundary problem for a non-linear elliptic equation of the second order in a bounded domain with a degenerate boundary. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 42–44. (Russian)

The author calls attention to the fact that for a wide class of non-linear elliptic equations (arising from certain variational problems) boundary values can be prescribed on boundary components of arbitrary dimensionality. Theorem: Let Ω be a bounded domain in E^n whose boundary S may consist of components of dimension

$n-1, \cdots, 1, 0$ (but is such that for Ω and S the embedding theorem holds [see S. L. Sobolev, *Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike*, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565; p. 64]), and let the functional

$$J(u) = \int_\Omega F\left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)d\Omega$$

satisfy the following conditions: (A) For some $p>n$, $J(u)$ is lower semicontinuous in $W_p^{(1)}$, and it follows from $J(u_n)\to 0$ that $\|u_n\|_{L_p^{(1)}}\to 0$ [see Sobolev, loc. cit. p. 52, for definitions]; (B) $J(u)$ is uniformly convex. Then for an arbitrary function $\varphi(s)$ defined on S and admissible with respect to $W_p^{(1)}$ the minimum of $J(u)$ in $W_p^{(1)}(\varphi)$ is assumed for some $u_0 \in W_p^{(1)}(\varphi)$. Example: $n=2$, $\Omega=\{(x,y)|0<x^2+y^2<1\}$; $\varphi=1$, $x^2+y^2=1$; $\varphi=0$, $x^2+y^2=0$;

$$J(u) = \int_\Omega \{\alpha(u_x^2+u_y^2)^2+\beta(u_x^2+u_y^2)\}dxdy \quad (\alpha>0, \beta>0).$$

P. Henrici (Los Angeles, Calif.)

2822:

Krylov, A. L. The first boundary value problem for some parabolic quasilinear equations in Banach space. Dokl. Akad. Nauk SSSR **124** (1959), 264–267. (Russian)

In the preceding note [reviewed above] the author studied solutions for a partial differential equation for a "non-linear circular membrane". In this note he studies the analogous "heat equation" in an n-dimensional region and states sufficient conditions for existence and uniqueness of a solution. *M. M. Day* (Urbana, Ill.)

2823:

Magenes, Enrico. Sui problemi al contorno per i sistemi di equazioni differenziali lineari ellittici di ordine qualunque. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 25–45.

The author gives an exposition of the questions and recent results (and, at the same time, an outline for further investigations—here mostly referring to unpublished work) in the field of the boundary value problems of differential equations of an order higher than the second and of systems of differential equations, both of elliptic type. *J. C. C. Nitsche* (Minneapolis, Minn.)

2824:

Koševlev, A. I. On boundedness in L_p of the derivatives of the solutions of elliptic equations and systems. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 323–325. (Russian)

The author considers an elliptic system L (in the sense of Petrovsky) of order $2m$ with coefficients in $C^m(G)$ for some domain G of R^n, and considers solutions in $W^{2m,p}(G)$ (the family of vector-valued functions in $L^p(G)$ all of whose distribution derivatives of order $\leq 2m$ also lie in $L^p(G)$) of the equation $Lu=f$. If G' is a sub-domain of G with positive distance from its boundary and if L is strongly elliptic, the author's basic result states that there exists a constant $C(G')$ such that for $j\leq 2m$, $\|D^j u\|_{L^p(G')} \leq C(G')\|f\|_{L^p(G)}$. Weaker results are obtained for general elliptic systems. {Results of the same general type were announced about the same time as the present note by Nirenberg [Comm. Pure Appl. Math. **9** (1956), 509–529; MR **19**, 962] and the reviewer [ibid. 351–361; MR **19**, 862].} A very brief indication is given of the method of proof. *F. Browder* (New Haven, Conn.)

2825:

Naĭmark, B. M. Completeness of the system of proper and associated functions of strongly elliptic systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 198–201. (Russian)

Generalizing results announced for second-order elliptic equations by M. V. Keldyš [same Dokl. **77** (1951), 11–14; MR **12**, 835] and established for elliptic equations of arbitrary order by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 433–439; MR **14**, 984], the present note states and sketches the proof of the following theorem: Let $L=B+\mu K+T$ be a strongly elliptic system of order $2m$ with smooth coefficients, B formally self-adjoint, K skew-adjoint, T of lower order. If the characteristic functions u_j of L of order j with characteristic value λ are defined recursively by $(L-\lambda)u_j=u_{j-1}$, with all the u_j satisfying the null Dirichlet boundary conditions of order m on the boundary of the domain G on which L and the u_j are defined, then the closed linear span in $L_2(G)$ (of vector-valued functions) of all the characteristic functions of L is all of $L_2(G)$ if the parameter μ is sufficiently small. The proof given is similar to that of the reviewer's paper cited above. {Reviewer's note: The proof as given does not seem to prove the assertion for L of the form $B+\mu K+T$ but rather of the more special form $B+\mu(K+T)$. A modification of the argument using the relations between the L_2-norms of derivatives of various orders as applied in the reviewer's paper [loc. cit.] would yield the conclusion as stated without difficulty.}

F. Browder (New Haven, Conn.)

2826:

Thomée, Vidar. Existence proofs for mixed problems for hyperbolic differential equations in two independent variables by means of the continuity method. Math. Scand. **6** (1958), 5–32.

The paper considers, in the first instance, a system of first-order linear hyperbolic equations in two independent variables, defined on a certain type of simply-connected region with a piecewise-smooth boundary. In an earlier paper [Math. Scand. **5** (1957), 93–113; see the review MR **19**, 1059, for a more precise description of the hyperbolic operator and the simply-connected region] the author derived a priori estimates for the solutions of the mixed boundary problem. Here he bases an existence proof on these estimates. The method of continuity involves treating the general problem of a perturbation of a simpler special problem, by imbedding both in a one-parameter family. As in the earlier paper, the results for first-order systems apply to give parallel results for a single higher-order equation.

R. W. McKelvey (Boulder, Colo.)

2827:

Campbell, L. Lorne. Solution of a mixed problem for a hyperbolic differential equation by Riemann's method. Acta Math. **100** (1958), 23–43.

This paper treats a mixed problem for a single hyperbolic equation of order n in two independent variables. Data are given on an initial curve and on a single boundary

curve which intersects it. The problem is a special case of one treated by Campbell and Robinson [Proc. London Math. Soc. (3) **5** (1955), 127–147; MR **16**, 1116] and by V. Thomée [see preceding review], where a more complicated boundary was allowed. The present paper adapts a method which was previously used by F. Rellich [Math. Ann. **103** (1930), 247–278] to solve the initial value problem for this equation.

R. W. McKelvey (Boulder, Colo.)

2828:

Wang, Guang-ying. Cauchy problem and singular Cauchy problem for systems of linear partial differential equations. Sci. Record (N.S.) **1** (1957), 323–328.

In this note the author considers the system of partial differential equations of first order

$$Ku_t = A^\alpha u_\alpha + Bu + f, \tag{1}$$

for a vector function $u=(u_1, \cdots, u_l)$ in $n+1$ independent variables $t, x_1, \cdots, x_n$. The matrices A^α and K are assumed symmetric, and K positive definite. For initial values assigned on $t=0$, the uniqueness of the solution is proved by the energy integral, and the existence proof is indicated along the line of Cauchy-Kowalewski and Schauder, to obtain C_1 solutions, assuming that the coefficients of the equations are sufficiently smooth. The case of non-negative K which may degenerate to become indefinite is also discussed. System (1) has been extensively treated by Friedrichs in his 1954 paper on symmetric hyperbolic linear differential equations [Comm. Pure Appl. Math. **7** (1954), 345–392; MR **16**, 44]. Apparently the author was not aware of this work.

Yu Why Chen (New York, N.Y.)

2829:

Ovsyannikov, L. V. Groups and invariant-group solutions of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 439–442. (Russian)

The author describes a method of finding particular solutions of systems of partial differential equations, especially solutions invariant under a group which acts on the space and transforms solutions into solutions. Adiabatic motion of a gas serves as an example [cf. the author's paper, Dokl. Akad. Nauk SSSR **111** (1956), 47–49; MR **19**, 89].

G. A. Hunt (Ithaca, N.Y.)

2830:

Kinoshita, Nobuo; and Mura, Toshio. On Stefan's problem. Sûgaku **8** (1956/57), 216–218. (Japanese)

On a two-dimensional space with coordinates t and x, given a system of partial differential equations with two unknowns, χ and u: (i) χ depends only upon t with its derivative continuous; (ii) $\chi(0)=0$; (iii) $u_{xx}=u_t$ for $0<x<\chi(t)$, $0<t<\gamma$; (iv) $u=0$ and $u_x=-d\chi/dt$ at $x=\chi(t)$, $0<t<\gamma$; and (v) $-u_x+hu=f(t)$ at $x=0$, $0<t<\gamma$, where h is a given constant and $f(t)$ is a given function of t. The authors call this a Stefan's type problem of the two-dimensional case and claim that if f is a function satisfying a Lipschitz condition in t, then there is a uniquely determined solution. The proof depends upon a skillful use of Tychonoff's fixed-point theorem but the lack of details makes it difficult for the reviewer to follow.

H. Yamabe (Minneapolis, Minn.)

2831:

Huet, Denise. Phénomènes de perturbation singulière. II. C. R. Acad. Sci. Paris **248** (1959), 58–60.

Suite d'une note antérieure [mêmes C. R. **247** (1958), 2273–2276; MR **21** #758]. Étude du comportement lorsque $\varepsilon>0$, $\varepsilon\to 0$ de la solution u_ε du problème mixte

$$(\varepsilon A(t)+B(t))u_\varepsilon+\frac{d}{dt}u_\varepsilon=f(t),$$

$u_\varepsilon(0)=0$, $u_\varepsilon(t)\in$ domaine de $A(t)$ pour $t>0$, et $A(t)$ étant d'ordre plus grand que $B(t)$. Variantes diverses. Lien avec la théorie des semi-groupes lorsque A et B ne dépendent pas de t.

J. L. Lions (Nancy)

POTENTIAL THEORY

See also 2811, 2849, 2990a-c, 3010, 3011, 3012, 3013, 3014, 3015.

2832:

Brelot, M. Quelques développements récents sur le problème de Dirichlet. Abh. Math. Sem. Univ. Hamburg **23** (1959), 48–59.

Expository lecture.

J. L. Doob (Urbana, Ill.)

2833:

Gask, H.; and Roos, J.-E. A simple model from potential theory. Nordisk Mat. Tidskr. **6** (1958), 5–20, 55. (Swedish. English summary)

The authors illustrate modern methods in potential theory (especially the use of Hilbert space) by considering the following simple model: Fix $d>0$ and let X be the set of points nd, $n=0, \pm 1, \pm 2, \cdots$. Fix a function $\psi(r)>0$ and decreasing to 0 as r goes to ∞. Define a function φ on X by $\varphi(x)=\sum_{y\geq|x|} d\cdot\psi(y)$ (y in X), the series being assumed convergent. By a distribution of charges is meant a function e on X vanishing outside a finite set. Then e_x denotes the amount of charge at the point x. The potential due to the distribution e is defined as the function P on X with $P(x)=\sum_y \varphi(x-y)e_y$. Thus the function φ itself gives the potential due to a unit-charge at the origin. The above definition of φ in terms of ψ is of course motivated by interpreting $-ef\psi(r)$ as the force between two charges e and f at distance r, and regarding $\varphi(x)$ as the amount of work needed to move a unit charge from $+\infty$ to x, if $x>0$. From its definition, it follows that φ is even and convex for $x>0$. The total energy $I(e)$ of a distribution e is now defined as $\frac{1}{2}\sum_{x,y}\varphi(x-y)e_xe_y=\frac{1}{2}\sum_x P(x)e_x$, where P is the potential due to e. Theorem 1: Every non-vanishing distribution has positive energy.

Let now R be the space of all distributions. It has an obvious structure as linear space. R is made into an inner-product space by setting $(e,f)=\frac{1}{2}\sum\varphi(x-y)e_xf_y$. Thus $(e,e)=I(e)$. Using the geometry of inner-product spaces the following results are proved: Let "conductor" mean a bounded subset of X. Theorem: Given t disjoint conductors L_k and for each a number l_k: then there exists a unique distribution, the so-called equilibrium distribution, having minimal energy consistent with total charge l_k on each L_k. Theorem: The potential of the equilibrium distribution is constant on each conductor. Theorem: If the distribution $e>0$, then its potential P takes its maxi-

mum value somewhere in the support of e, i.e. in the set where $e_x \neq 0$. Theorem: The equilibrium-potential of a positive charge on a conductor L is positive. A final theorem gives a criterion in order that equilibrium distributions be concentrated on the boundaries of conductors. *J. Wermer* (Cambridge, Mass.)

2834:

Matsushita, Shin-ichi. On some function spaces concerning Dirichlet's problem. Proc. Japan Acad. **34** (1958), 76–78.

Remarques sur le caractère fonctionnel de la solution du problème de Dirichlet et le rôle des points-frontière irréguliers. *J. Deny* (Palaiseau)

2835:

Matsushita, Shin-ichi. On the foundation of balayage theory. J. Inst. Polytech. Osaka City Univ. Ser. A **9** (1958), 59–86.

Nouvelle version, avec généralisations et rectifications, d'un travail antérieur [Proc. Japan Acad. **31** (1955), 643–647; **32** (1956), 29–34, 125–130; MR **18**, 29; voir aussi J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 57–90; MR **19**, 953]. L'idée originale de l'auteur consiste à associer aux points réguliers d'un ensemble fermé (i.e., points en lesquels E n'est pas effilé) les points extrémaux d'un ensemble compact et convexe convenable. Le théorème de Krein et Milman conduit alors immédiatement à l'existence d'au moins un balayage. Ensuite on redémontre, en utilisant de nombreux résultats de la théorie newtonienne fine (principe de domination pour les potentiels d'énergie finie, théorèmes concernant les suites de potentiels, etc.) l'existence d'un balayage minimum unique, la mesure balayée chargeant seulement l'ensemble des points réguliers. On considère également le balayage sur un ouvert. On se place dans un cadre plus général que celui de la théorie newtonienne, mais sans donner d'autre exemple que celui des potentiels de Green. *J. Deny* (Palaiseau)

2836:

Lowdenslager, David B. Potential theory in bounded symmetric homogeneous complex domains. Ann. of Math. (2) **67** (1958), 467–484.

On établit: soit G un groupe compact agissant sur l'espace séparable X, laissant invariant x, et agissant transitivement sur un ensemble B. Alors si F est un espace vectoriel de fonctions continues invariant par G, contenant les constantes, et si

$$|f(x)| \leq \max_{b \in B} |f(b)|$$

pour toute $f \in F$, on a une représentation intégrale:

$$f(x) = \mu(f) = \int_B f(b)db = \int_G f(gb)dg$$

à partir de l'intégrale de Haar.

On cherche à préciser cette représentation pour les fonctions harmoniques de certains domaines homogènes, le laplacien Δ étant formé à partir du ds^2 invariant. Une formule de Poisson est donnée pour le domaine de C^n (x_k+iy_k) qui est une tube:

$$y_n > (y_1^2 + \cdots + y_{n-1}^2)^{1/2} \quad (-\infty < x_k < +\infty).$$

L'ensemble B joue le rôle de frontière "distinguée" quand D est un domaine homogène non dégénéré: si f est une donnée continue sur B, il existe dans D une solution de $\Delta u = 0$ qui tend vers la valeur $f(x)$, $x \in B$, quand on tend vers x sur un chemin qui est précisé. *P. Lelong* (Paris)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 2647, 2659, 3043, 3132.

2837:

Soldatov, M. A. Solution of linear difference equations with linear coefficients. Mat. Sb. (N.S.) **47 (89)** (1959), 221–236. (Russian)

The author considers the linear difference equation $\sum_k (a_k x + b_k) y(x+h_k) = 0$, which by means of the Laplace transform can be reduced to a first order linear differential equation. The paper is devoted to a detailed discussion of the Laplace inverse of the solution of this equation. *R. Bellman* (Santa Monica, Calif.)

2838:

Berg, Lothar. Behandlung einer Funktionalgleichung. Wiss. Z. Univ. Rostock. Math.-Nat. Reihe **5** (1955/56), 367–372.

This paper deals with the following two infinite systems of equations

$$x_{\nu-1} - x_\nu = cx_{2\nu}; \quad x_{\nu-1} - x_\nu = x_{2(\nu-1)} \quad (\nu = 2, 3, 4, \cdots),$$

where c denotes a positive constant. In order to find a particular solution for each of these systems the author applies the "method of sections", i.e., one takes in each of these two systems the first n equations ($\nu = 2, 3, \cdots, n+1$). In these n equations one keeps only the terms in which the unknowns $x_1, x_2, \cdots, x_{n+1}$ occur and one cancels all other terms. After solving the resulting finite system of equations one lets $n \to \infty$.

In the case of the equations $x_{\nu-1} - x_\nu = cx_\nu$, the author gives for every n an explicit solution of the finite system. By using a diagonal method a solution of the corresponding infinite system can be found. *J. Popken* (Amsterdam)

2839:

Kuczma, Marek. On convex solutions of the functional equation $g[\alpha(x)] - g(x) = \varphi(x)$. Publ. Math. Debrecen **6** (1959), 40–47.

The following theorem is proved. Let $\alpha(x)$ be concave, continuous and not less than x in $[a, \infty]$. Let $\varphi(x)$ be continuous in $[a, \infty]$ and both positive and increasing for sufficiently large x. In addition, let $\lim_{n\to\infty} [\varphi(a_n) - \varphi(a_{n-1})] = 0$, where $a_0 = \alpha(a)$, $a_n = \alpha(a_{n-1})$. Then there exists at most one convex function $g(x)$ satisfying the functional equation $g[\alpha(x)] - g(x) = \varphi(x)$ and assuming a prescribed value at $x = a$. Unless $\lim_{x\to\infty} \alpha(x)/x = 1$, there is no solution. This theorem may be regarded as a generalization of the fact that the only convex solution of the equation $g(x+1) - g(x) = \log x$ for which $g(0) = 0$ is the logarithm of the gamma function. *J. W. Green* (Los Angeles, Calif.)

SEQUENCES, SERIES, SUMMABILITY

See also 2753.

2840:

Schönhofer, A.; und Zuser, K. Eine Verallgemeinerung der Fibonaccischen Zahlenfolge. Elem. Math. **14** (1959), 38–39.

The authors generalize the Fibonacci sequence, defined by $a_0=0$, $a_1=1$, $a_n=a_{n-2}+a_{n-1}$ for $n\geqq 2$, to sequences defined as follows: for an integer $k\geqq 2$, let $a_0, a_1, \cdots, a_{k-1}$ be arbitrary non-negative real numbers of which at least one is positive, and let

$$a_n = a_{n-k}+a_{n-k+1}+\cdots+a_{n-1}$$

for $n\geqq k$. The authors' main result is that

$$\lim_{n\to\infty}\frac{a_{n+r}}{a_n}=x_1^r$$

for any fixed non-negative integer r, where x_1 is the unique positive root of the equation

$$x^k-x^{k-1}-\cdots-x-1=0.$$

A. Sklar (Chicago, Ill.)

2841:

Diananda, P. H. Extensions of an inequality of H. S. Shapiro. Amer. Math. Monthly **66** (1959), 489–491.

Shapiro's inequality

$$\sum_{i=1}^{n}\frac{x_i}{x_{i+1}+x_{i+2}}\geqq\frac{n}{2}\quad(x_{n+j}=x_j>0 \text{ for all } j)$$

has been proved true for $n<6$ and false for $n=20$ [same Monthly **63** (1956), 191–192]. The inequality is here considerably generalized, e.g. to

$$\sum\frac{x_i}{x_{i+1}+\cdots+x_{i+m}}\geqq\frac{n}{m},$$

which is proved true (so that Shapiro's inequality is true for $n=6$) if n divides any one of the four numbers $m+2$, $2m$, $2m+1$, $2m+2$. Two more general results are given.

S. H. Gould (Providence, R.I.)

2842:

Dawson, David F. Convergence of continued fractions of Stieltjes type. Proc. Amer. Math. Soc. **10** (1959), 12–17.

The author gives two new results on the convergence of the continued fraction

$$\text{(A)}\qquad\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}+\cdots,$$

where $b=\{b_p\}_{p=1}^{\infty}$ is a complex number sequence. (1) Let $\{g_p\}_{p=0}^{\infty}$ be the sequence of approximants of (A). If $\{g_{2p-1}\}$ [$\{g_{2p}\}$] converges absolutely and $\{g_{2p}\}$ [$\{g_{2p-1}\}$] converges, then (A) converges if and only if b satisfies one of the following conditions: (i) the series $\sum|b_{2p-1}|$ [the series $\sum|b_{2p}|$] diverges; (ii) $\limsup|b_2+b_4+\cdots+b_{2p}|=\infty$ [$\limsup|b_1+b_3+\cdots+b_{2p-1}|=\infty$]. (2) Let $g_p=C_p/D_p$. If $\sum|b_{2p+1}||D_{2p}|^2$ converges and $\{g_p\}_{p=k}^{\infty}$ is a bounded complex number sequence for some k, then (A) converges if and only if b satisfies the condition

$$\lim|b_2+b_4+\cdots+b_{2p}|=\infty.$$

E. Frank (Chicago, Ill.)

2843:

Wuyts-Torfs, M. On a regular generalization of the Euler summation-method. Simon Stevin **32** (1958), 170–175. (Dutch)

The Euler summation method, which replaces the sequence $\{s_n\}$ ($n=0, 1, 2, \cdots$) of partial sums of an infinite series $u_0+u_1+\cdots$ by the sequence $\{\sigma_n\}$,

$$\sigma_n=\sum_{k=0}^{n}\binom{n}{k}r^k(1-r)^{n-k}s_k$$

($n=0, 1, 2, \cdots$; r constant, $0<r<1$) was studied by Agnew [Amer. J. Math. **66** (1944), 313–338; MR **6**, 46] for the general case in which r is a complex constant, and generalized by replacing r by a sequence $r_0, r_1, r_2, \cdots$, of real numbers, putting

$$\sigma_n=\sum_{k=0}^{n}\binom{n}{k}r_n^k(1-r_n)^{n-k}s_k\quad(n=0, 1, 2, \cdots).$$

The author shows that this summability method is regular (i.e., leaves invariant the sums of convergent series) if and only if the sequence $r_0, r_1, r_2, \cdots$ satisfies the conditions (1) $n\alpha_n$ is bounded, and (2) $\lim_{n\to\infty}n\beta_n=+\infty$, where

$$\alpha_n=|r_n|+|1-r_n|-1,\quad\beta_n=1-|1-r_n|\qquad(n=0, 1, 2, \cdots).$$

L. M. Blumenthal (Columbia, Mo.)

2844:

Russell, Dennis C. Convolution of Nörlund summability methods. Proc. London Math. Soc. (3) **9** (1959), 1–20.

Convolution A^*B of two matrix methods A, B has been defined by Vermes [J. Analyse Math. **2** (1952), 160–177; MR **14**, 745]. Using Toeplitz' theorem, the author obtains necessary and sufficient conditions for the inclusion (*) $(N, p_n)^*(N, q_n)\supset(N, t_n)$. He applies this result to several special cases. We quote the following. Write $\{p_n\}\in M$ if $p_n>0$ and $p_n/p_{n-1}\leqq p_{n+1}/p_n\leqq 1$. Let $t_n=p_0q_n+\cdots+p_nq_0$. Then the relation (*) holds if $\{p_n\}, \{q_n\}\in M$. It holds also if $\{P_n\}, \{Q_n\}\in M$, $T_n>0$, $T_1+\cdots+T_n\to\infty$, where P_n, Q_n, T_n stand for the sums from 0 to n of the corresponding p_k, q_k, t_k. For the Cesàro methods with $\rho, \sigma, \tau>-1$, the relation $(C, \rho)^*(C, \sigma)\supset(C, \tau)$ holds if and only if $\tau\leqq\rho+1$, $\tau\leqq\sigma+1$, $\tau\leqq\rho+\sigma$. This improves somewhat a result of Moustafa [J. London Math. Soc. **30** (1955), 85–100; MR **16**, 465]. *G. G. Lorentz* (Syracuse, N.Y.)

2845:

Bhatt, Shri Nivas. A Tauberian theorem for absolute Riesz summability. Indian J. Math. **1**, no. 1, 29–32. (1958).

The series $\sum a_n$ is said to be absolutely summable (R, λ, k) if the function $A_\lambda^k(w)=\sum_{\lambda_n\leqq w}(w-\lambda_n)^k a_n$, where $\{\lambda_n\}$ is positive, steadily increasing and divergent and $k\geqq 0$, is of bounded variation in (A, ∞) for some $A>0$. The author establishes the following theorem which improves a previous result of T. Pati [Math. Z. **61** (1954), 75–78; MR **16**, 465]. If $\sum a_n$ is absolutely summable (R, λ, k), $k>0$, $\{\lambda_m^{-1}\sum_1^m\lambda_n a_n\}$ is a sequence of bounded variation and $\{\lambda_n/\lambda_{n+1}\}$ is also of bounded variation, then $\sum a_n$ is absolutely convergent.

A. G. Azpeitia (Amherst, Mass.)

2846:

Gordon, Basil. On a Tauberian theorem of Landau. Proc. Amer. Math. Soc. **9** (1958), 693–696.

The author proves the following theorem: Let $A(x)$ be nondecreasing, $T(x)=\sum_{n\le x}A(x/n)$. Assume that $T(x)=x\log x+bx+o(x/\log x)$. Then $A(x)/x\to 1$ as $x\to\infty$. The author's proof is elementary. [Cf. A. E. Ingham, J. London Math. Soc. **20** (1945), 171–180; MR **8**, 147; H. Shapiro, Proc. Amer. Math. Soc. **1** (1950), 346–348; MR **12**, 80; and a paper by J. Karamata, *Colloque sur la théorie des suites, Bruxelles, 18–20 Dec., 1957*, pp. 12–31, Centre Belge Recherches Math., Gauthier-Villars, Paris, Établ. Ceuterick, Louvain, 1958; MR **20** #7173.]

P. Erdős (Haifa)

2847:

Koshti, M. E.; and Kabe, D. G. Modification to Euler's constant. J. Karnatak Univ. **1** (1956), no. 1, 176–182.

The Euler sequence $\sum_1^n (n^{-1}-\log n)$ is generalized by the first author to

$$\sum_0^n \left\{\frac{1}{1+nx}-\frac{1}{x}\log(1+nx)\right\}$$

and by the second author to

$$\sum_0^n \left\{\frac{1}{n+x}-\log(n+x)\right\},$$

with some study of the resulting functions of x.

APPROXIMATIONS AND EXPANSIONS

See also 2806.

2848:

Korobov, N. M. Approximate evaluation of repeated integrals. Dokl. Akad. Nauk SSSR **124** (1959), 1207–1210. (Russian)

Let $p>3$ be prime. For integral $n\ge 1$ define $H(z_1,\cdots,z_n)$ as follows.

$$H(z_1,\cdots,z_n)=\sum_{k=1}^{p-1}[1-2\ln(2\sin\pi\{kz_1/p\})]\cdots \times[1-2\ln(2\sin\pi\{kz_n/p\})],$$

where $z_1,\cdots,z_n$ are arbitrary integers of the interval $1\le z\le p-1$ and $\{kz_\nu/p\}$ is the fractional part of the number kz_ν/p.

The integers $a_1,a_2,\cdots$ will be called optimal coefficients if $1\le a_1\le p-1$ and for given $a_1,\cdots,a_\nu$ $(\nu\ge 1)$ the magnitude of $a_{\nu+1}$ equals one of the values of z $(1\le z\le p-1)$ for which $H(a_1,\cdots,a_\nu,z)$ achieves its minimum.

We consider the function $f(x_1,\cdots,x_s)$ defined on the unit s-dimensional cube $(s\ge 1)$ which has absolutely convergent Fourier series:

$$f(x_1,\cdots,x_s)=\sum_{m_1,\cdots,m_s=-\infty}^{\infty} C(m_1,\cdots,m_s)e^{2\pi i(m_1x_1+\cdots+m_sx_s)}.$$

Theorem 1: If there exist constants $\alpha>1$ and $C=C(s)$ such that the Fourier coefficients of the function $f(x_1,\cdots,x_s)$ fulfill the condition

$$|C(m_1,\cdots,m_s)|\le \frac{C}{[(|m_1|+1)\cdots(|m_s|+1)]^\alpha},$$

then, for each collection of optimal coefficients $a_1,a_2,\cdots$, for $\xi_\nu(k)=\{ka_\nu/p\}$ $(\nu=1,2,\cdots,s)$, and for any $\varepsilon>0$, the following estimate is valid:

$$\int_0^1\cdots\int_0^1 f(x_1,\cdots,x_s)\,dx_1\cdots dx_s - \frac{1}{p}\sum_{k=1}^{p} f[\xi_1(k),\cdots,\xi_s(k)]=O\left(\frac{1}{p^{\alpha-\varepsilon}}\right).$$

Theorem 2: For each $\alpha>1$ there exists a function $f(x_1,\cdots,x_s)$ the Fourier coefficients of which satisfy the conditions

$$|C(m_1,\cdots,m_s)|\le \frac{C}{[(|m_1|+1)\cdots(|m_s|+1)]^\alpha}$$

such that for any numbers $a_1,a_2,\cdots$ with $\xi_\nu(k)=\{ka_\nu/p\}$ $(\nu=1,2,\cdots,s)$ we have

$$\left|\int_0^1\cdots\int_0^1 f(x_1,\cdots,x_s)dx_1\cdots dx_s - p^{-1}\sum f[\xi_1(k),\cdots,\xi_s(k)]\right|>p^{-\alpha}.$$

A. H. Stroud (Madison, Wis.)

2849:

Elianu, Jean. Sulla rappresentazione delle funzioni analitiche di più variabili reali. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 21–25, 146–151.

Cioranescu [Bull. Soc. Math. France **65** (1937), 41–52 and other papers] and others have established the following result. An analytic function u of the (real) variables $x_1,\cdots,x_n$ which is regular in a neighborhood of the origin can be expanded in the form $u=\phi_0+\phi_1r^2+\phi_2r^4+\cdots$, where $r^2=x_1^2+\cdots+x_n^2$ and the ϕ_k are harmonic functions of $x_1,\cdots,x_n$. The present paper contains a (complex variable) proof of this result along the following lines. Assuming first that the expansion exists and can be differentiated term-by-term, the author determines each ϕ_k in terms of the values of $\Delta^k u$ (Δ is Laplace's operator) on the characteristic cone $r^2=0$ through the origin. He then estimates the ϕ_k and shows that the expansion is convergent and may be differentiated any number of times. Finally he proves that an analytic function v for which $\Delta^k v$, $k=0,1,2,\cdots$ vanishes along the characteristic cone $r^2=0$, vanishes identically; and deduces that the expansion constructed by him represents u.

A. Erdélyi (Pasadena, Calif.)

2850:

Akopyan, S. A.; and Nersesyan, A. B. Some integrodifferential operators and expansions in series analogous to Schlömilch series. Akad. Nauk Armyan. SSR. Dokl. **27** (1958), 201–207. (Russian. Armenian summary)

Given one set of biorthogonal functions $\{\phi_n(x),\psi_n(x)\}$ $(n=0,1,\cdots)$ over the interval (a,b) it is shown that by use of some integrodifferential operators it is possible to obtain a new system of biorthogonal functions $\{\Phi_n(x),\Psi_n(x)\}$ $(n=0,1,\cdots)$ on (a,b). Starting with the Bessel functions $J_\nu(j_{n,\nu}x)$, this procedure yields a new system $\{\Psi_n(x)\}$ and so enables the author to establish the expansion theorem: let the function $f(x)$, given in $(0,1)$,

be such that a certain operator applied to $f(x)$ exists; then if

$$a_n = \int_0^1 f(t)\Psi_n(t)dt \quad (n = 1, 2, \cdots),$$

where Ψ_n results from applying the operator to $J_\nu(j_{n,\nu}x)$, then $f(x)$ may be expanded in the form

$$f(x) = \sum_{n=1}^{\infty} a_n\left(\frac{2}{j_{n,\nu}}\right)^{\alpha} x^{\nu+\alpha}J_{\nu+\alpha}(j_{n,\nu}x) \quad (0 \leq \alpha < 1),$$

uniformly on $[0, \delta] \subset [0, \Delta)$ $(0 < \Delta < 1)$.

C. G. Maple (Ames, Iowa)

2851:

Malliavin, P.; et Mandelbrojt, S. Sur l'équivalence de deux problèmes de la théorie constructive des fonctions. Ann. Sci. École Norm. Sup. (3) **75** (1958), 49–56.

Let E be a closed set on the real line which is the union of closed intervals. The authors link the problem whether the polynomials with weight $p(x) > 0$ $(x \in E)$ are dense in the space of continuous functions on E vanishing at infinity with the question whether there are functions regular in the complement of E in the z-plane and satisfying various order conditions there. The complete statement of their results is too long for reproduction. This paper is a continuation of researches by S. Mandelbrojt [*General theorems of closure*, Rice Inst. Pamphlet, special issue, 1951; MR **13**, 540] and J. Horváth [Math. Scand. **2** (1954), 83–90; MR **15**, 955].

W. H. J. Fuchs (Ithaca, N.Y.)

2852:

Timan, M. F. Interrelation between total and partial best approximation in the mean of functions of several variables. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 24–26. (Russian)

Consider a function $f(x_1, x_2)$ of period 2π, belonging to L^p; let $E_{n_1,n_2}(f)$ denote the best approximation to f by trigonometric polynomials of degree at most n_1 in x_1 and n_2 in x_2. For almost every x_2, define $E_{n_1}(f; x_2)$ as the best approximation to f as a function of x_1, and then put $E_{n_1,\infty}(f) = \|E_{n_1}(f; x_2)\|_{L^p}$; this is a "partial best approximation". Similarly $E_{n_2,\infty}(f)$ arises from considering f as a function of x_1. There are corresponding definitions for functions of more variables. The author states the following theorem and carries out the proof for the case $k=2$: for $1 < p < \infty$,

$$E_{n_1,\cdots,n_k}(f) \leq C_p \min\{E_{n_{\nu_1},\cdots,n_{\nu_i},\infty}(f) + E_{n_{\nu_{i+1}},\cdots,n_{\nu_k},\infty}(f)\},$$

where the minimum is taken over $\nu_m = 1, 2, \cdots, k$; $m = 1, 2, \cdots, i$; while for $p = 1$ or ∞,

$$E_{n_1,\cdots,n_k}(f) \leq C \min\{E_{n_{\nu_1},\cdots,n_{\nu_i},\infty}(f) + E_{n_{\nu_{i+1}},\cdots,n_{\nu_k},\infty}(f) \log n_{\nu_1} \cdots \log n_{\nu_i}\},$$

with $\nu_m = 1, 2, \cdots, k$; $m = 1, 2, \cdots, i$; $i \leq [k/2]$.

R. P. Boas, Jr. (Evanston, Ill.)

2853:

Timan, M. F. On the relationship between total and partial best approximations of a function of several variables. Dokl. Akad. Nauk SSSR **124** (1959), 527–528. (Russian)

Results such as

$$E_{n_1n_2}(f) \leq C\left\{E_{m\infty}(f) + E_{\infty n_2}(f) \log \frac{n_1}{n_1 - m + 1}\right\}, \quad 1 \leq m \leq n_1.$$

Here $f(x_1, x_2)$ is a continuous 2π-periodic function, $E_{n_1n_2}(f)$ is its degree of uniform approximation by trigonometric polynomials of order $\leq n_1$ in x_1 and of order $\leq n_2$ in x_2. Many misprints. See also M. F. Timan [paper reviewed above] and S. Bernšteĭn [Trudy Mat. Inst. Steklov **38**, pp. 24–29, Izdat. Akad. Nauk SSSR, Moscow, 1951; MR **13**, 842].

G. G. Lorentz (Syracuse, N.Y.)

2854:

Remez, E. Ya. Sur le problème du minimax algébrique pour un système fini de fonctions linéaires. II. Ukrain. Mat. Ž. **10** (1958), 289–298. (Russian. French summary)

Part I appeared in Ukrain. Mat. Ž. **10** (1958), no. 2, 178–192 [MR **20** #3409]. For notation and terminology, see the review cited. The author here considers the problem of finding the subset of $\mathfrak{N}_0$ for which the solution of the minimax problem satisfies certain additional conditions.

E. Hewitt (Seattle, Wash.)

FOURIER ANALYSIS

See also 2740, 2750, 2773, 2812, 2906, 2917.

2855:

Gloden, Raoul-François. Séries de Fourier et polynômes. Bull. Soc. Roy. Sci. Liège **28** (1959), 141–149.

The Fourier series for the function f such that $f(x) = x$, $-\pi \leq x < \pi$, is integrated termwise k times. Expressions are obtained for the resulting polynomials.

P. Civin (Eugene, Ore.)

2856:

Salem, Raphael. Recherches récentes sur l'unicité du développement trigonométrique. Rend. Mat. e Appl. (5) **17** (1958), 319–326.

The author gives a summary of known relations of certain symmetric perfect sets to the uniqueness or multiplicity of trigonometric series.

P. Civin (Eugene, Ore.)

2857:

Nikolaev, V. F. Polynomial operations in certain spaces. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 639–641. (Russian)

Let E_t denote spaces consisting of 2π-periodical functions that contain all trigonometric polynomials as a dense set, that are invariant under translation, with norm

$$\|f(x+t)\| = \|f(x)\|.$$

A_n will be called a polynomial operator of order n in E_1 if $A_n(E_1) \subseteq T_n \subseteq E_2$, where T_n is the set of trigonometric polynomials of order n. Definition: $A \in S$ will mean that

$A(f(t+x), x) = A(f(x), x+t)$. Let $S_n (\in S)$ take f into its Fourier partial series in T_n. Then

$$\int_0^{2\pi} A_n(f(x+t), x-t)dt = \int_0^{2\pi} A_n(S_n f(x+t), x-t)dt.$$

Of all A_n that agree on T_n with a $B_n \in S$, the smallest one is B_n itself. If we vary the basic space E, the smallest norm that a $B_n \in S$ can have is the one when $E = C$, where C is an E-space consisting of continuous functions with the uniform norm. *František Wolf* (Berkeley, Calif.)

2858:

Lamperti, John. A note on conjugate functions. Proc. Amer. Math. Soc. **10** (1959), 71–76.

Let $\phi(t)$ be a non-negative, convex function defined for all $t \geq 0$ with $\phi(0) = 0$ and $\phi(2t) \leq K\phi(t)$ for some constant K. Let ϕ belong to any of the following classes. A: $\phi'(t)$ is concave, and $\phi(t^\theta)$ is convex for some $\theta < 1$; B: $\phi'(t)$ is convex, and $\phi(t^{1-\theta})$ is concave for some $\theta < 1$; C: $\phi = \phi_1 + \phi_2$, where $\phi_1 \in$ A and $\phi_2 \in$ B; D: $0 < a \leq \phi(t)/\phi_1(t) \leq b < \infty$ for all large t, where $\phi_1 \in$ C; E: $\phi(t) = t^p L(t)$, where $1 < p \neq 2$, and $L(t)$ is slowly varying. It is proved that if $f \in L_\phi$ (i.e., $I(f) = \int_0^{2\pi} \phi(|f(x)|)dx < \infty$), then its conjugate $\bar{f} \in L_\phi$ and there is a constant C such that $\|\bar{f}\|_\phi \leq C\|f\|_\phi$, where $\|f\|_\phi$ denotes a norm in the Orlicz space L_ϕ. The basic idea of the proof is that of Stein's proof for the case L_p. It is also pointed out that for the classes A–E the following relations hold: A $\cup$ B $\subset$ C $\subset$ D; E $\subset$ D.

W. A. J. Luxemburg (Pasadena, Calif.)

2859:

O'Shea, Siobhan. On a divergent trigonometrical series given by Steinhaus. Proc. Amer. Math. Soc. **10** (1959), 68–70.

Steinhaus gave the series

$$(*) \qquad \sum_{n=2}^{\infty} (\log n)^{-1} \cos n(x - \log\log n)$$

as an example of an everywhere divergent trigonometric series with coefficients tending to zero. The author shows that a condition on $u(n) \uparrow \infty$ and $c_n \downarrow 0$ sufficient that $\sum_{n=1}^{\infty} c_n \cos(nu(n)) \cos nx$ should diverge for all real x is the existence of a sequence of positive integers p_n such that

$$\limsup_{n\to\infty} (n+p_n)[u(n+p_n) - u(n)] < \tfrac{1}{2}$$

and

$$\liminf_{n\to\infty} \sum_{m=n+1}^{n+p_n} c_m > 0.$$

The special case $u(n) = \log\log n$, $c_n = (\log n)^{-1}$, $n \geq 2$, gives the "cosine part" of (*). The condition is also sufficient that $\sum_{n=1}^{\infty} c_n \sin(nu(n)) \sin nx$ should diverge for all $x \not\equiv 0 \pmod{\pi}$. *R. R. Goldberg* (Evanston, Ill.)

2860:

Pati, T. On the absolute Nörlund summability of a Fourier series. J. London Math. Soc. **34** (1959), 153–160.

Let (p_n) be a positive non-increasing sequence and let $P_n = p_0 + p_1 + \cdots + p_n \to \infty$. The series $\sum a_n$ is said to be $|N, p_n|$ summable when the sequence $t_n = \sum_{\nu=0}^{n} p_{n-\nu} s_\nu / P_n$ is of bounded variation. If $f(t)$ is of bounded variation in the interval $(0, \pi)$ and the sequences

$$\{(n+1)p_n/P_n\} \quad \text{and} \quad \left\{ \sum_{\nu=0}^{n} P_\nu/(\nu+1)P_n \right\}$$

are also, then the Fourier series of f is $|N, p_n|$ summable at $t = 0$. If $p_n = \binom{n+\alpha-1}{\alpha-1}$, then $|N, p_n|$ summability reduces to $|C, \alpha|$ summability and the theorem reduces to Bosanquet's theorem. *S. Izumi* (Sapporo)

2861:

Kahane, Jean-Pierre; et Katznelson, Yitzhak. Sur la réciproque du théorème de Wiener-Lévy. C. R. Acad. Sci. Paris **248** (1959), 1279–1281.

If E is a compact subset of $(0, 2\pi)$, let $A(E)$ be the set of all functions on E which are representable as sums of absolutely convergent trigonometric series. Let F be a function defined on a compact convex set D in the plane, such that $F(f) \in A(E)$ whenever $f \in A(E)$ and $f(E) \subset D$. The authors prove that if E contains arbitrarily long arithmetic progressions (for instance, if E contains the sequence $\{1/n\}$), then every F with the above property is real-analytic in a neighborhood of D. For the case in which E is an interval, this result was proved earlier by Helson and Kahane [same C. R. **247** (1958), 626–628; MR **20** #4737] and Katznelson [ibid., 404–406; MR **20** #4152]. It is also pointed out that there are perfect sets E such that every continuous function on E belongs to $A(E)$; in that case, every continuous function F on D will do.

W. Rudin (Madison, Wis.)

2862:

Prasad, B. N.; and Siddiqi, J. A. On the Gibbs phenomenon for the Nörlund means. Indian J. Math. **1**, no. 1, 21–27 (1958).

The Nörlund (N, p) transformation of the sequence $\{s_n(x)\}$ is the sequence $\{\sum_{\nu=0}^{n} p_{n-\nu} s_\nu(x) / \sum_{\nu=0}^{n} p_\nu\}$. The authors show that if $p_n > 0$ is monotonic decreasing, then the (N, p) transformation of any sequence of partial sums of the Fourier series of a function of $L(0, 2\pi)$ does not exhibit the Gibbs phenomenon. They also establish a condition between two Nörlund transformations under which the Gibbs phenomenon for a sequence $\{S_n(x)\}$ bounded in a neighbourhood of x_0 can occur at x_0 for one of the transformations only if it also occurs for the other.

H. G. Eggleston (London)

2863:

Weston, J. D. An inequality for the Fourier coefficients of a non-negative function. Amer. Math. Monthly **66** (1959), 223–224.

Let $g(\theta)$ be a non-negative function with complex Fourier coefficients c_n over $(-\pi, \pi)$. Then $c_0^2 + c_0|c_{2n}| \geq 2|c_n|^2$. The author was unaware that the three-dimensional analogue is well known to crystallographers under the name of the Harker-Kasper inequality; cf. MR **12**, 496.

R. P. Boas, Jr. (Evanston, Ill.)

2864:

Edwards, R. E. Comments on Wiener's Tauberian theorems. J. London Math. Soc. **33** (1958), 462–466.

In his book *The Fourier integral and certain of its*

applications [Cambridge University Press, 1933, reprinted by Dover, New York, 1959; MR **20** #6634], N. Wiener gives two theorems (Theorems 6 and 7, p. 75), which he says are variants of one theorem, and which he calls the general Tauberian theorem. However, Wiener's proofs of these two results are quite independent of each other and both are long. In the present paper, the author presents a very simple argument showing that these two theorems of Wiener are in fact equivalent. He also points out that Wiener's Theorem 7 can be proved directly just as his Theorem 6 is proved from a study of the group algebra $L_1(R)$. One considers the commutative Banach algebra M consisting of all continuous complex-valued functions f on R such that

$$\|f\| = \sum_{n=-\infty}^{+\infty} \max(|f(x)| : n \leq x \leq n+1) < \infty.$$

The linear operations are pointwise and multiplication is convolution. One shows that the multiplicative linear functionals on M are all of the form $f \to \int_{-\infty}^{+\infty} f(t)e^{-iyt}dt$, and then applies the technique described, for example, in L. H. Loomis, *An introduction to abstract harmonic analysis* [Van Nostrand, Toronto-New York-London, 1953; MR **14**, 883], pp. 147–148. *E. Hewitt* (Seattle, Wash.)

2865:

Greenstein, D. S. Derivative manifolds and Taylor series in the mean. Trans. Amer. Math. Soc. **90** (1959), 312–322.

Supposons que $f \in C^\infty(-\infty, \infty)$, et que pour un certain $p \geq 1$, $f^{(n)} \in L^p(-\infty, \infty)$ $(n \geq 0)$. Désignons par $\delta(f)$ le sous-espace vectoriel fermé de L^p engendré par $\{f^{(n)}\}$ $(n \geq 0)$, et par $\tau(f)$ celui engendré par les translatées de f. On soit que $\delta(f) \subset \tau(f)$. En désignant par $L^p\{M_n\}$, où $\{M_n\}$ est une suite positive, la classe des fonctions de $C^\infty(-\infty, \infty)$ pour lesquelles on a $\|f^{(n)}\|_p \leq M_n$ $(n \geq 0)$, Mandelbrojt a démontré qu'une condition nécessaire et suffisante (n. et s.) pour que pour chaque $f \in L^p\{M_n\}$ on ait $\delta(f) = \tau(f)$ est que la classe $C\{M_n\}$ soit quasi analytique. Lorsque $p = 2$, l'auteur indique une condition n. et s. pour $\delta(f) = \tau(f)$ portant directement sur f. Ainsi, en désignant par F la transformée de Fourier de f, une condition n. et s. pour que $\delta(f) = \tau(f)$, est que la fonction $\psi(t) = \int_{-\infty}^{t} |F(x)|^2 dx$ soit la solution d'un problème des moments de Hamburger à solution unique. L'auteur indique ensuite des conditions pour que les translatées de f soient représentées par les "séries de Taylor en moyenne", c'est-à-dire pour qu'on puisse écrire

$$f(x+h) = \text{l.e.m.}_{n=\infty} \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(\nu)}(x).$$

L'auteur en tire des conclusions pour les transformées de Fourier dans le domaine complexe.

S. Mandelbrojt (Paris)

2866:

Koosis, Paul. Sur la non-totalité de certaines suites d'exponentielles sur des intervalles assez longs. Ann. Sci. École Norm. Sup. (3) **75** (1958), 125–152.

The author considers those increasing sequences $\{\lambda_n\}$ of positive numbers that have a finite density (i.e. $\lim n/\lambda_n = D < \infty$) and the associated collection $\{\exp(\pm i\lambda_n X)\}$ of exponential functions. Such a collection is total on each interval of length less than $2\pi D$ in the sense that if dm is a measure supported by such an interval and

$$\int \exp(\pm i\lambda_n X) dm(X) \equiv 0$$

then $dm \equiv 0$. The author deals with the question: given a sequence $\{\lambda_n\}$ with finite density D, what further conditions on the regularity of the distribution of the λ_n guarantee that the functions $\{\exp(\pm i\lambda_n X)\}$ are not total on any interval of length exceeding $2\pi D$? Kahane [Ann. Inst. Fourier. Grenoble 8 (1958), 273–275] has recently shown that additional conditions are indeed needed, by producing a sequence $\{\lambda_n\}$ with $D = 0$ but such that $\{\exp(\pm i\lambda_n X)\}$ are total on any finite interval whatsoever.

Denoting by $C(Z) = \prod (1 - Z^2/\lambda_n^2)$ the associated canonical product, the author introduces the notion of redressability: $\{\lambda_n\}$ is redressable if, for each $\varepsilon > 0$, there is a sequence of density $\leq \varepsilon$ (whose associated canonical product is denoted by $C_\varepsilon(Z)$) such that $C(X)C_\varepsilon(X)$ is majorized, for X real, by a polynomial in X. He proves (theorem 7) that $\{\exp(\pm i\lambda_n X)\}$ is non-total on each interval of length exceeding $2\pi D$ if and only if $\{\lambda_n\}$ is redressable.

The main part of the paper is concerned with applying elementary estimates to $|C(X)|$ in order to derive simple sufficient conditions that $C(X)$ be majorized by a polynomial in X, and also that $\{\lambda_n\}$ be redressable. Examples are given showing that none of these sufficient conditions is also necessary. Along the same lines, roughly comparable necessary conditions are given, but they fall short of being sufficient.

The author points out that the methods used do not seem powerful enough to give a complete solution of the problem, but the results, both positive and negative, are of considerable interest in themselves. For example (theorem 11, corollary) if $n(t) - Dt = O(t^\alpha)$ for some $\alpha < 1$ as $t \to \infty$, where $n(t)$ is the number of λ_n not exceeding t, then $\{\lambda_n\}$ is redressable, but the converse is false.

There is some overlap in methods and results between the present paper and a paper of Redheffer [Trans. Amer. Math. Soc. **77** (1954), 32–61; MR **16**, 915].

L. A. Rubel (Princeton, N.J.)

2867:

Mandelbrojt, S. Composition theorems. Rice Inst. Pamphlet **45** (1958/59), no. 3, v + 63 pp.

This is a continuation of the earlier work of the author, contained in *Séries adhérentes* [Gauthier-Villars, Paris, 1952; MR **14**, 542] and *General theorems of closure* [Rice Inst. Pamphlet, special issue, 1951; MR **13**, 540]. Chapter I contains a review of some of the concepts which are basic in this subject: various density functions for sets of positive integers λ_n, adherent series, the Fourier-Carleman transform and the spectrum $\sigma(f)$ of a function f defined by means of this transform. Certain classes of functions are introduced; these classes consist of infinitely differentiable functions on the line, and are characterized by conditions such as these: $f^{(n)} \in L^p$, $\|f^{(n)}\|_p \leq M_n$, $\sigma(f) \subset E$, $f^{(\lambda_n)}(0) = f(0) = 0$ $(n \geq 1)$. This class is called $L^p(\{M_n\}, \{\lambda_n\}, E)$.

The statement "$\{M_n\}$ is associated (p) with E and $\{\lambda_n\}$" is defined to mean the following: there exists no function $\Phi \not\equiv 0$, holomorphic and single-valued in the complement

of E (in the upper half-plane if E is the whole real line), such that $|\Phi(z)| \leq A|y|^{(1/p)-1}$ and

$$\left|\Phi(z) - \sum_{n=1}^{m} \frac{d_n}{z^{\lambda_n+1}}\right| \leq A\,\frac{M_{q+1}}{|z|^{q+1}}\,|y|^{(1/p)-1}$$

for $\lambda_m \leq q < \lambda_{m+1}$ and some choice of complex numbers $d_1, \cdots, d_m$.

Chapter II, the longest of three, contains a number of theorems whose conclusion is that at least one of the classes under consideration is trivial (i.e., contains only the function $f \equiv 0$). Their general spirit may be illustrated by theorem 8: Suppose (1) $\{\log M_n\}$ and $\{\log M_n'\}$ are convex functions of n, (2) $\{\lambda_n\}$ and $\{\lambda_n'\}$ are increasing sequences of even positive integers, (3) N is the complement of these two sequences with respect to the set of all even positive integers, (4) E_1 and E_2 are closed symmetric sets on the line, neither containing the origin, (5) $\{M_nM_n'\}$ is associated (2) with $E_1 \otimes E_2$ (the set of all $a \cdot b$ with $a \in E_1$, $b \in E_2$) and the sequence N. Then one of the classes $L^2(\{M_n\}, \{\lambda_n\}, E_1)$, $L^2(\{M_n'\}, \{\lambda_n'\}, E_2)$ contains no non-trivial even function.

In theorem 10 the conclusion is that all functions belonging to a certain class of this type are periodic. Toward the end of this chapter, some conditions are established which imply that a sequence $\{M_n\}$ is associated (p) with a set E and a sequence $\{\lambda_n\}$.

Chapter III contains a discussion of various problems related to the problem of "associating a sequence with a set" in the above sense. *W. Rudin* (Madison, Wis.)

2868:

Reiter, H. Contributions to harmonic analysis. IV. Math. Ann. **135** (1958), 467–476.

[Pour les trois premières parties, voir Acta Math. **96** (1956), 253–263; Math. Ann. **133** (1957), 298–302; J. London Math. Soc. **32** (1957), 477–483; MR **19**, 290; **20** #1170ab.] L'auteur étudie la sous-algèbre de l'algèbre $\mathcal{F}L^1(R^n)$ ($\mathcal{F}$ = transformée de Fourier; $n \geq 3$) formée des fonctions ne dépendant que de la distance à l'origine (caractères, théorème de Wiener, idéaux maximaux et primaires; éléments f n'appartenant pas à l'adhérence de l'idéal engendré par f^2). *L. Schwartz* (Paris)

2869:

Essén, Matts. On homomorphisms and orthogonal systems. Ark. Mat. **3** (1958), 505–510.

The problem: Let G be a compact group, D a subgroup of the group of continuous homomorphisms T of G onto G. Let A_D be the functions f on G such that

$$\int_G f(Tx)\overline{f(Sx)}dx = \begin{cases} 0, & f(Tx) \not\equiv f(Sx), \\ 1, & f(Tx) \equiv f(Sx). \end{cases}$$

Question: When is A_D essentially a subset of $\hat{G}$, the character group of G?

Results: I. Let G be the dual of the discrete real line, D the homomorphisms T that reduce to real multiplication when x in G is real: $Tx = ax$, a real, nonzero. Let A_0 be the class of functions f on G such that

$$\mathfrak{M}\{f(ax)\overline{f(bx)}\} = \lim \frac{1}{2T}\int_{-T}^{T} f(ax)\overline{f(bx)}dx = \begin{cases} 0, & \text{if } a \neq b,\ ab \neq 0, \\ 1, & \text{if } a = b = 1; \end{cases}$$

and let $A_1 \subset A_0$ be those functions in A_0 having Fourier series $\sum_{k=1}^{\infty} a_k e^{i\lambda_k x}$, $\lambda_k > 0$. (Note: $f \in A_0 \Rightarrow M\{f\} = 0$.) If $f \in A_1$ and $M(x) = \sum_{k=1}^{\infty} |a_k|\lambda_k x$ exists for all x, and if $\{\lambda_k\}$ has at most one of 0 and ∞ as limit point, then at most one $a_k \neq 0$. (The restriction to A_1 is inessential.) II. Let G be the circle, D the group of all continuous homomorphisms of G onto G, $D = \{T | Tx = nx(2\pi)\}$, and A_2 the class of functions f in A_D such that the Fourier series of f is of the form $\sum_{k=1}^{\infty} a_k e^{ikx}$. If $f \in A_2 \cap C^{(\infty)}$, then at most one $a_k \neq 0$. (In I, "$M(x)$ finite" may not be weakened to "$M(x)$ finite for $x > b$ or for $x < b$". In II, "$f \in A_2 \cap C^{(\infty)}$" may not be weakened to "$f \in A_2 \cap C^{(n)}$, $n < \infty$".) III. If $F = \{f(nx)\}_{n=-\infty}^{\infty}$, then F is complete in $L^2(G)$ if and only if $f = ae^{ix} + be^{-ix}$, where $|a|^2 + |b|^2 = 1$, $\mathrm{Re}(ab) = 0$.

B. R. Gelbaum (Minneapolis, Minn.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 2837, 2850.

2870:

Wintner, Aurel. On the kernel of Lindelöf's representation of analytic prolongation. Math. Scand. **6** (1958), 281–288.

For $0 < \alpha < 1$ and also for the "limiting case" $\alpha = 1$, the entire function $\sum_{n=1}^{\infty} n^{-\alpha n} z^{n-1}$ is, with all its derivatives, positive on the real axis $-\infty < x < \infty$, and admits a representation $\int_0^{\infty} e^{xt} d\mu_\alpha(t)$ with $d\mu_\alpha(t) \geq 0$.

S. Bochner (Princeton, N.J.)

2871:

Weston, J. D. An extension of the Laplace-transform calculus. Rend. Circ. Mat. Palermo (2) **6** (1957), 325–333.

The algebraic theory of operational calculus by Mikusiński [Studia Math. **11** (1949), 41–70; MR **12**, 189; *Rachunek operatorów*, Polskie Towarzystwo Matematyczne, Warsaw, 1953 (the book is now also available in German and English); MR **16**, 243] has little room for the analytic niceties of the classical Laplace transformation. The basic reason is not that elements such as impulses are locally bad, but that functions such as $\exp(e^t)$ just grow too fast as $t \to \infty$. The present author seeks to re-establish the Laplace transformation as the basis for operational calculus, while retaining certain advantages of Mikusiński's theory. The resulting theory is simpler but less general than the latter, and seems rather comparable in scope, although not in method, to Laplace transform theory based on distributions.

For the purpose of this review a function x will be called decent if for some c the product $x(t)e^{-ct}$ is integrable over $(0, \infty)$. Decent functions x have Laplace transforms $\bar{x}$ in the classical sense; these are holomorphic in some half-plane $\mathrm{Re}\, s > c$. The author calls a function p perfect if p is infinitely differentiable on $(0, \infty)$, with $p^{(k)}(0+) = 0$, and $p^{(k)}$ decent ($k = 0, 1, \cdots$). The convolution of a perfect and a decent function is always perfect.

The objects of the author's operational calculus are the perfect operators A defined as follows. A is called perfect if it takes every perfect function p into a perfect function Ap, and if the Laplace transform of Ap has the form $\bar{A}(s)\bar{p}(s)$ where $\bar{A}$ is holomorphic in a half-plane $\mathrm{Re}\, s > c$ and independent of p. The function $\bar{A}$ is called the

Laplace transform of A. The correspondence $A \leftrightarrow \bar{A}$ is an isomorphism. Thus the perfect operators form a commutative ring $\mathfrak{A}$ without zero divisors, and with unit element. $\mathfrak{A}$ contains the differential operator D, the shift operators (delta functions) I_u with $u \geqq 0$, and all operators whose Laplace transforms are rational. Every decent function x determines a perfect operator X defined by convolution: $Xp = x * p$. The Laplace transform of X is the same as that of x, and hence one can identify X with x, thus embedding the decent functions in $\mathfrak{A}$. In $\mathfrak{A}$ every decent function x has a generalized derivative DX.

J. Korevaar (Dordrecht)

2872:

Ditkin, V. A. On the theory of operational calculus. Dokl. Akad. Nauk SSSR **123** (1958), 395–396. (Russian)

The author states as his aim the construction, for Mikusiński's operational calculus, of an analog of the classical Laplace transformation [cf. the preceding review]. Let L denote the ring, with respect to the operations of addition and convolution, of the functions on $(0, \infty)$ which are integrable over all finite intervals $(0, A)$. Mikusiński's field of operators is the quotient field $R(L)$. The author obtains a complicated-looking ring M which is isomorphic to L, and thereby a field $R(M)$ which is isomorphic to $R(L)$. The ring M is essentially the following. The elements have the form

$$\bar{F} = (\bar{F}_1, \bar{F}_2, \cdots) = \left(\int_0^1 f(t)e^{-st}dt, \int_0^2 f(t)e^{-st}dt, \cdots \right),$$

where $f \in L$. Addition in M is defined in the usual way; the product $\bar{F}\bar{G}$ is given by $(\bar{F}\bar{G})_n \equiv \bar{F}_n\bar{G}_n \bmod J_n$. Here J_n consists of the Laplace transforms of all functions (which have one and) which vanish on $(0, n)$. The author finally states that the field $R(M)$ can be investigated by analytic function methods.

J. Korevaar (Dordrecht)

2873:

Mikusiński, J. Sur les notions de distribution et d'opérateur. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 737–741.

The author compares his operators [Studia Math. **11** (1949), 41–70; MR **12**, 189] with Schwartz's distributions on $(-\infty, \infty)$. An operator and a distribution are said to be equal if there exists a sequence of continuous functions which converges to the operator in the operator sense and to the distribution in the distribution sense. The author proves that (i) a distribution is an operator if and only if its support is bounded below; (ii) the operator $\exp(s^{1/2}) = 1/f$, where $f(t) = 0$ for $t \leqq 0$, $f(t) = \frac{1}{2}\pi^{-1/2}t^{-3/2}\exp\{-1/(4t)\}$ for $t > 0$, is not a distribution. [Closely related results were indicated by L. Schwartz in his review of Mikusiński's basic paper.] More generally the author determines the character of the operator $\exp(-\lambda s^\alpha)$ for $0 < \alpha < 1$ and all complex λ.

J. Korevaar (Dordrecht)

2874:

Butzer, P. L. Die Anwendung des Operatorenkalküls von Jan Mikusiński auf lineare Integralgleichungen vom Faltungstypus. Arch. Rational Mech. Anal. **2** (1958), 114–128.

The paper emphasizes the simplicity of Mikusiński's operational calculus [Studia Math. **11** (1949), 41–70; MR **12**, 189] as well as its theoretical advantages over the Laplace transformation when dealing with large functions. After summarizing Mikusiński's calculus the author proves results on the sum of the series $\sum_{n=1}^{\infty} f^n$ under various hypotheses on the size of f. He then proceeds to solve a large variety of special cases of the equation

$$\alpha_n f^{(n)}(t) + \cdots + \alpha_0 f(t) + \int_0^t k(t-u)f(u)du = g(t),$$

where the α's may be zero. The special cases are selected so as to encounter a minimum of technical difficulties; solutions are obtained in closed form. The author also considers a simple quadratic convolution equation, systems of Volterra equations, and special cases of Abel's singular equations.

J. Korevaar (Dordrecht)

2875:

Weston, J. D. Operational calculus and generalized functions. Proc. Roy. Soc. London Ser. A **250** (1959), 460–471.

L'A. construit un calcul opérationnel analogue à celui de Mikusiński, mais en utilisant un autre procédé d'extension. Soit X l'espace vectoriel des fonctions indéfiniment dérivables sur $(0, \infty)$ et tendant vers 0 quand $t \to 0$. L'A. considère comme espace opérationnel l'ensemble $\mathfrak{A}$ des opérateurs linéaires sur X, permutables avec la convolution, opérateurs qu'il appelle "parfaits".

Tous les opérateurs parfaits, en particulier la dérivation et son inverse, s'étendent à cet espace. La transformation de Laplace fournit un isomorphisme avec une algèbre de fonctions de variable complexe.

G. Marinescu (Bucharest)

2876:

Koizumi, Sumiyuki. On the singular integrals. III. Proc. Japan Acad. **34** (1958), 594–598.

[For preceding parts, see same Proc. **34** (1958), 193–198, 235–240; MR **20** #5402, #5403.]

Results on Hilbert's operator that are due to L. H. Loomis [Bull. Amer. Math. Soc. **52** (1946), 1082–1086; MR **8**, 377] and to the reviewer [J. London Math. Soc. **18** (1943), 66–71; MR **5**, 96] are considered from a general point of view. Some theorems on the interpolation of an operation are needed; they are based on the work of Calderón, Marcinkiewicz and Zygmund, and are applied to the operator $\tilde{f}(x) = \lim \int_{|x-t|>\eta} f(t)(x-t)^{-1}dt$ $(\eta \to 0)$. Then results of the following type are obtained:

$$\int_{-\infty}^{\infty} \frac{\varphi(|\tilde{f}|)}{1+|x|^\alpha}\,dx \leqq A \int_{-\infty}^{\infty} \frac{\varphi(|f|)}{1+|x|^\alpha}\,dx,$$

$$\int_{-\infty}^{\infty} \frac{|\tilde{f}|}{1+|x|^\alpha}\,dx \leqq A \int_{-\infty}^{\infty} \frac{|f| \log^+[(1+x^2)|f|]}{1+|x|^\alpha}\,dx + B,$$

where $0 \leqq \alpha < 1$, and $\varphi(x)$ $(x > 0)$, e.g. $\varphi(x) = x^p$ $(p > 1)$, is a function satisfying certain conditions; and a counterexample is indicated for the cases $p > 1$, $\alpha \geqq 1$ and for $p = 1$, $\alpha > 1$.

Analogous results hold for discrete sequences of numbers and for Dirichlet's singular integral.

H. Kober (Birmingham)

2877:

Vilenkin, N. Ya. The matrix elements of irreducible unitary representations of a group of Lobachevsky space

motions and the generalized Fock-Mehler transformations. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 219–222. (Russian)

Let G be the group of all unimodular linear transformations leaving invariant the quadratic form

$$x_1^2+x_2^2+\cdots+x_n^2-x_{n+1}^2.$$

Let Ω_n be the subgroup carrying 0, 0, $\cdots$, 0, 1 into itself and the plane $x_{n+1}=0$ into itself. When $n=2$ the decomposition of an "arbitrary" function constant on the $\Omega_n:\Omega_n$ double cosets into zonal spherical functions leads to an integral transform previously studied by Mehler [Math. Ann. **18** (1881), 161–194] and Fock [Dokl. Akad. Nauk SSSR **39** (1943), 253–256; MR **5**, 181]. In this note the author observes that one obtains a series of similar transforms by letting $n=3, 4, 5, \cdots$. He announces an L^2 theory of these transforms and describes and discusses a number of related identities involving special functions.

G. W. Mackey (Cambridge, Mass.)

2878:

Flett, T. M. Some theorems on fractional integrals. Proc. Cambridge Philos. Soc. **55** (1959), 31–50.

Let $\phi(z)$ be regular in $|z|<1$, let $\phi(z)=\sum_1^\infty c_n z^n$, and for any real α let $\phi_\alpha(z)=\sum_1^\infty (in)^{-\alpha}c_n z^n$. The author proves many interesting results, some idea of which may be gained from the following very incomplete selection. Let

$$g_{k,\alpha}^*(\theta)=\left\{\int_0^1 \frac{|\phi_\alpha(\rho e^{i\theta})-\phi_\alpha(\rho^2 e^{i\theta})|^k}{(1-\rho)^{1+k\alpha}}\,d\rho\right\}^{1/k}.$$

It is shown that if $k\geq 2$, $\lambda>0$, and $\alpha<1$ then $\|g_{k,\alpha}^*\|_\lambda \leq B\|\phi\|_\lambda$, while if $1<k\leq 2$, $\lambda>1$, and $\alpha>-1$ then $\|\phi\|_\lambda \leq B\|g_{k,\alpha}^*\|_\lambda$. Suppose again that $1<k\leq 2$, $\lambda>1$ and $\alpha\geq 0$, and that there exists a function $\chi(\theta)$ such that the integral

$$I=\int_{-\pi}^{\pi} d\theta\left\{\int_0^1 \frac{|\chi(\theta)-\phi_\alpha(\rho e^{i\theta})|^k}{(1-\rho)^{1+k\alpha}}\,d\rho\right\}^{\lambda/k}$$

is finite. Then $\phi_\alpha(e^{i\theta})=\lim_{\rho\to 1-}\phi_\alpha(\rho e^{i\theta})$ exists and is equal to $\chi(\theta)$ for almost all θ; moreover we also have $\|\phi\|_\lambda \leq BI$. These researches are related to a paper of the reviewer [Amer. J. Math. **75** (1953), 513–546; MR **15**, 119], to which the author brings a number of valuable corrections. They are also related to work of G. Sunouchi [Tokyo Math. J. (2) **9** (1957), 307–317; MR **20** #7191].

I. I. Hirschman, Jr. (St. Louis, Mo.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 2874, 3130.

2879:

★**Гахов, Ф. Д. Краевые задачи.** [Gahov, F. D. Boundary problems.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 543 pp. 16.90 rubles.

This book is devoted to a study of the Riemann and the Hilbert boundary value problems and their application to the solutions of singular integral equations with Cauchy and Hilbert kernels, respectively. In this regard it covers the same material as the book by N. I. Muskhelishvili [*Singulyarnye integral'nye uravneniya*, OGIZ, Moscow-Leningrad, 1946; translated by J. R. M. Radok as *Singular integral equations*, Noordhoff, Groningen, 1953; MR 8, 586; 15, 434], omitting, however, any treatment of the applications. By emphasizing fundamentals and forsaking in many cases the complete generality characteristic of Muskhelishvili's book, the author achieves a high level of exposition in what will undoubtedly come to be regarded as the most readable textbook in the field. The reader's background need not go beyond a modest course in complex variable theory and a knowledge of the basic fundamentals of the Fredholm theory of integral equations, and even here the requisite theorems are all carefully summarized before they are used. At the close of each of the text's seven chapters one finds a page or two of "Historical information", followed by extensive sets of problems (with answers).

Chapter I treats integrals of Cauchy type and presents the all-important formulae of Plemelj relating, relative to a given contour, the left and right limiting values of a Cauchy type integral in terms of its density and its principal value on the contour. The author, however, credits these formulae to Yu. V. Sohotskiĭ and refers to them throughout the book as the formulae of Sohotskiĭ. His historical note states that Sohotskiĭ first derived these relations in 1873 in his doctoral dissertation. Chapter II presents the Riemann boundary value problem and its solution for both simple and multiply-connected regions (referred to as the Hilbert problem by Muskhelishvili). This work is then used in chapter III in solving characteristic type singular integral equations with Cauchy kernels. Three operator methods are then presented for reducing the complete singular integral equation with Cauchy kernel to a Fredholm equation. Chapter IV treats the Hilbert problem (the Riemann-Hilbert problem in the terminology of Muskhelishvili) and then employs the results in solving singular integral equations with Hilbert kernels.

In chapter V the Riemann and Hilbert boundary problems are generalized to allow derivatives in the boundary conditions; e.g., in the Hilbert problem so generalized one seeks a function $\Phi(z)=u(x,y)+iv(x,y)$, analytic in a region D^+ with a simple, smooth boundary contour L, satisfying on the contour the condition

$$\sum_{k=0}^{m}\left[a_k(s)\frac{d^k u}{ds^k}+b_k(s)\frac{d^k v}{ds^k}\right]=f(s),$$

where the a_k, b_k, $f(s)$ are real and continuous on L, and a_m, b_m, $f(s)$ satisfy a Holder condition. Chapters VI and VII are concerned with the two boundary value problems and corresponding singular integral equations when discontinuous coefficients are permitted along the boundary contours and the contours are not closed.

An extensive list of references is given, including the most important non-Russian works. The book could be described as a pure mathematician's (suitably abridged) version of Muskhelishvili's book.

J. F. Heyda (Cincinnati, Ohio)

2880:

Zaripov, R. Kh. Systems of singular integral convolution equations. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 20–23. (Russian)

2881:
Kim, E. I. Solution of a certain class of singular integral equations with line integrals. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 24–27. (Russian)

2882:
Kim, E. I. On a certain class of singular integral equations. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 268–271. (Russian)

2883:
Ivanov, V. V. The use of the method of moments and the "mixed" method for an approximate solution of singular integral equations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 945–948. (Russian)

2884:
Germogenova, T. A. Bounded solutions of an inhomogeneous integral equation given on a semi-infinite interval and having its kernel dependent on the difference of the arguments. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 23–26. (Russian)

2885:
Mysovskih, I. P. Computation of the eigenvalues of integral equations by means of iterated kernels. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 45–48. (Russian)

2886:
Rybin, P. P. Convergence of series obtained in solving non-linear integral equations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 458–461. (Russian)

2887:
Rozovskiĭ, M. I. The problem of Cauchy for a partial integro-differential equation in unbounded space. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3 (75), 369–376. (Russian)

2888:
Sokoloff, G. D. Sur l'application de la méthode des corrections fonctionnelles moyennes aux équations intégrales non linéaires. Ukrain. Mat. Ž. **9** (1957), 394–412. (Russian. French summary)

2889:
Zaripov, R. H. Systems of complete singular integral equations of convolution type. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 429–432. (Russian)

Some of the results of an earlier paper [#2880 above] are extended by the addition of completely continuous terms, and the reduction to Fredholm equations is described. No proofs are given.

D. C. Kleinecke (Livermore, Calif.)

2890:
Latter, Richard. Approximate solutions for a class of integral equations. Quart. Appl. Math. **16** (1958), 21–31.

In this paper, the author attempts an extension of the Wiener-Hopf technique to the integral equation

$$(*) \quad f(x) = g(x) + \lambda \int_0^a k(x-y) f(y)\,dy \quad (0 \leq x \leq a).$$

It is assumed that f, g, and $k \in L^2(-\infty, \infty)$, and that f, g are $O(e^{c|x|})$, $c < 1$, while $k(x) = O(e^{-|x|})$. If $f(x)$ is defined outside of $(0, a)$ by the right side of (*), and if f_+, f_a, and f_- are Fourier integrals of f restricted to (a, ∞), $(0, a)$, and $(-\infty, 0)$, respectively, the author obtains a pair of simultaneous integral equations for f_- and f_+, and an expression for the desired f_a in terms of f_+. These equations can be solved if the Fourier transform of k is rational [for a bibliography in this case see, e.g., K. S. Miller and L. A. Zadeh, Trans. I.R.E. **IT-2** (1956), 72–75]. An approximate solution is found in other cases by moving the contours in the two integral equations to bring out a finite number of poles of the functions involved, and solving an approximate system of linear equations. The validity of this process is made plausible, and some error estimates and illustrations are given.

H. O. Pollak (Murray Hill, N.J.)

2891:
Melzak, Z. A. Entire operators and functional equations. Proc. Amer. Math. Soc. **10** (1959), 438–447.

Proofs are given of theorems about the existence and uniqueness of solutions of the equation $\partial f(x,t)/\partial t = K[f(x,t)]$. An important example of this equation is the Boltzmann transport equation in applied mathematics.

D. ter Haar (Oxford)

2892:
Dragoş, L. Sur l'équation intégro-différentielle de Prandtl. Com. Acad. R. P. Romîne **8** (1958), 451–459. (Romanian. Russian and French summaries)

Making use of a paper by I. N. Vekua [Akad. Nauk SSSR. Prikl. Mat. Meh. **9** (1945), 143–145; MR **7**, 207] the author reduces the problem of integration of the Prandtl integro-differential equation to a problem of the Hilbert type in the complex plane, cut along the real axis from $-b$ to $+b$. Further, the problem is reduced to the integration of a differential equation, the coefficients of which are holomorphic functions in the complex plane excluding the segment $(-b, +b)$. Under certain restrictions on the circulation and the chord of the profile the above problems are solved, and the final solution of the Prandtl equation is obtained. The solution is illustrated for the case of an elliptic wing [see also I. Filimon, Acad. R. P. Romîne Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 381–385; MR **20** #2940].

E. Leimanis (Vancouver, B.C.)

2893:
Uzakov, Yu. K. Integro-differential equations of second order admitting a Lie group. Izv. Akad. Nauk UzSSR. Ser. Fiz.-Mat. **1958**, no. 4, 53–63. (Russian. Uzbek summary)

It is shown how to find the general form of H and K if the equation

$$H(x, y, y', y'') = \int_a^b K(x, y, y', y'', \xi, \eta, \eta', \eta'')\,d\xi$$

is to be invariant under the infinitesimal transformation $u(x, y)\partial/\partial x + v(x, y)\partial/\partial y$. Here η, η', η'' stand for $y(\xi)$, $y'(\xi)$, $y''(\xi)$.

G. A. Hunt (Ithaca, N.Y.)

FUNCTIONAL ANALYSIS

See also 2736, 2742, 2858, 2873, 2875.

2894:

Ficken, F. A. Some uses of linear spaces in analysis. Amer. Math. Monthly **66** (1959), 259–275.

An expository lecture on the progress in analysis that resulted from the application of algebraic and topological ideas to function spaces. The topics dealt with are vector spaces, differential equations, Banach spaces and Hilbert spaces. *L. Nachbin* (Rio de Janeiro)

2895a:

Bastiani, Andrée. Polyèdres convexes de dimension quelconque. C. R. Acad. Sci. Paris **247** (1958), 1943–1946.

2895b:

Bastiani, Andrée. Pyramides topologiques. Applications à l'analyse. C. R. Acad. Sci. Paris **248** (1959), 175–178.

The notion of infinite-dimensional simplex recently introduced by Choquet has proved to be of considerable utility in functional analysis. [See Choquet, C. R. Acad. Sci. Paris **243** (1956), 555–557, 736–737; MR **18**, 288; **19**, 399; Zbl. **71**, 107; also *Séminaire Bourbaki*, 9e année (1956/57), 2e éd., exp. 139, Secrétariat mathématique, Paris, 1959.] Now, by methods quite different from those of Choquet, Miss Bastiani initiates a theory of infinite-dimensional convex polyhedra and polyhedral cones ("pyramids") which seems also to be destined for an important role in the applications of infinite-dimensional geometry to problems of analysis. For a point x of a convex set C, the supporting cone of C at x is defined as the union of all rays from x through the various points of C. It is proved that a finite-dimensional convex cone C (which includes its vertex) is polyhedral if and only if, for each $x \in C$, the supporting cone of C at x is closed. [For extensions to nonconical convex sets, see Bastiani, *Séminaire de topologie et de géométrie différentielle dirigé par C. Ehresmann*, 1re année (1957/58), exp. 19, Secrétariat mathématique, Paris, 1959; also Klee, Acta. Math. **102** (1959), 79–107; MR **21** #4390.] In an arbitrary linear space E, a convex pyramid is defined as a convex cone with vertex 0, such that for each $x \in C$, the supporting cone of C at x is closed in the fine topology (the finest locally convex Hausdorff linear topology on E). A strict convex pyramid P is one such that the convex hull of $P \cup S$ is closed in the fine topology for each linear subspace S of E. Two theorems of the first note are as follows: (I) If E is of countable dimension, then a subset of E is a convex pyramid if and only if its intersection with each finite-dimensional subspace is a convex pyramid. (II) A strict convex pyramid is the intersection of its extreme supporting half-spaces. A simplicial pyramid is defined as the cone generated by a Hamel basis, and it is observed that if a convex pyramid P is contained in a simplicial pyramid, then P is the convex hull of its extreme rays. In the second note, there is proved an analogous result involving the notion of topological base, this being in the metrizable case merely a set which is total and is topologically free in the sense of Bourbaki: Suppose E is a Hausdorff linear space and S' is the convex cone generated by a topological base of E. Then if P is a convex pyramid contained in S', the closure of P is the closed convex hull of its extreme rays.

The "théorème fondamental" of the second note is as follows: Suppose S' is the convex cone generated by a topological base, S is the closure of S', H is a closed linear subspace of finite codimension, $C = H \cap S$, and $C' = H \cap S'$. Then C is the closure of C', C is a topological pyramid (closure of a convex pyramid) and is the closed convex hull of its extreme rays, which are also the extreme rays of C'. Further, the intersection of C with any finite family of closed (homogeneous) half-spaces is a topological pyramid C_1, the closure of the intersection C_1' of the half-spaces with C'. The set C_1 is the closed convex hull of its extreme rays, each of which is an extreme ray of C_1'. This leads to an extension of a theorem of Rosenbloom [Bull. Soc. Math. France **79** (1951), 1–58; **80** (1952), 183–215; MR **15**, 233]: Suppose S is as above, $f_1, \cdots, f_m$ are continuous linear functionals on E, and P is the set of all $x \in S$ for which $f_i x = a_i$ $(1 \leq i \leq k)$ and $f_i x \geq a_i$ $(k < i \leq m)$. Then P has at least one extreme point. If P intersects each line in a bounded segment, then P is the closed convex hull of its set of extreme points, each of which has at most m nonzero coordinates (in terms of the topological base in question). (For the space l, this was established by Rosenbloom, loc. cit.) The author indicates that there are many applications of the theorem, and outlines its application in the space of analytic functions.

The two notes contain some interesting results in addition to those stated above, but very little in the way of proofs. More details are available in the notes from the *Séminaire Ehresmann* referred to above, but it is to be hoped that Miss Bastiani will soon publish a detailed exposition in generally available form.

V. L. Klee, Jr. (Copenhagen)

2896:

Bastiani, Andrée; et Ehresmann, Charles. Sur les appuis d'une pyramide convexe et sur les polyèdres convexes sans sommet. C. R. Acad. Sci. Paris **248** (1959), 2695–2697.

This is a sequel to the two earlier notes by Miss Bastiani reviewed above, and communicates new results in her theory of infinite-dimensional convex polyhedra in a real linear space E. Consider a convex cone C (with vertex $0 \in C$) in E, and a linear subspace L of E. Then L is a support of C provided L contains, for each $x \in L \cap C$, the entire facet of x in C; and L is an extreme support provided L is a support which is actually the linear extension of $L \cap C$. The convex cone C is a pyramid provided the set $\operatorname{conv}(C \cup Rx)$ is closed (in the fine topology) for each $x \in C$. A pyramid P is strict provided $\operatorname{conv}(P \cup L)$ is closed for each subspace L of E, and proper provided $\operatorname{conv}(P \cup L)$ is closed whenever L is an extreme support of P. It was proved earlier that each strict pyramid is the intersection of half-spaces bounded by its extreme supporting hyperplanes. Here the authors announce that for each support L of a strict pyramid P there exists a supporting hyperplane H of P such that $P \cap H = P \cap L$. This strong result makes it reasonable to conjecture (as they do) that a strict pyramid must be the intersection of a finite family of half-spaces, and suggests the desirability of a less restrictive notion, such as that of a proper pyramid (introduced in the present note). They remark that while a finite-dimensional convex polyhedral

cone is a pyramid both strict and proper, an infinite-dimensional simplicial pyramid is proper but not strict. They state that if P is a proper pyramid in E, L' a subspace of E, and L an extreme support of $P \cap L'$ with $L \neq L'$, then there exists an extreme supporting hyperplane H of P such that $H \subset L$ and the intersection $H \cap L'$ is an extreme supporting hyperplane (in L') of the set $P \cap L'$. From this it follows that a proper pyramid is the intersection of half-spaces bounded by its extreme supporting hyperplanes.

In addition, the note contains various remarks and examples, complementing the earlier work and dealing especially with the notion of generalized simplicial pyramid. *V. L. Klee, Jr.* (Copenhagen)

2897:

Mehdi, M. R. Continuity of semi-norms on topological vector spaces. Studia Math. **18** (1959), 81–86.

A semi-norm on a real or complex vector space E is a real valued function $\rho(x)$ such that (a) $\rho(\lambda x) = |\lambda|\rho(x)$ for all scalars λ, (b) $\rho(x+y) \leq \rho(x) + \rho(y)$. It is shown that if E is a topological vector space of second category with finite semi-norm ρ and if the set $B = \{x: \rho(x) \leq 1\}$ is a Baire set in E, then ρ is continuous. This generalizes theorems of Bosanquet and Kestelman [Proc. London Math. Soc. (2) **45** (1939), 88–97] and Eberlein [Ann. of Math. (2) **47** (1946), 688–703; MR **8**, 279]. *R. E. Fullerton* (College Park, Md.)

2898:

Braunschweiger, Chris C. A geometric construction of the M-space conjugate to an L-space. Proc. Amer. Math. Soc. **10** (1959), 77–82.

A Banach lattice X is an M space if, in addition to the Banach lattice postulates, the following two conditions hold for X: (a) $x \wedge y = \theta$ implies $\|x+y\| = \|x-y\|$; (b) $x > \theta$, $y > \theta$ imply $\|x \vee y\| = \max(\|x\|, \|y\|)$. A Banach lattice is an L space if (a) is satisfied and also (b′) $x \geq \theta$, $y \geq \theta$ imply $\|x+y\| = \|x\| + \|y\|$. An element $e \in X$ is an F-unit if $e > \theta$ and $e \wedge x > \theta$ for every $x > \theta$. The following theorem is proved. Let X be an L space with F-unit e, partial ordering denoted by $>$, positive cone C, and unit sphere U. Define $S = (C-e) \cap (e-C)$, $p(x) = \inf\{a: x/a \in S,\ a > 0\}$ and $Y = \{aS: a > 0\}$. Then Y is an M space with unit element e, the same partial ordering, positive cone $C \cap Y$, norm $p(x)$ and unit sphere $S = \{x: p(x) \leq 1,\ x \in Y\} \subset U$. Also, there exists an isometry which is also an isomorphism preserving both algebraic and lattice operations between Y and the space L^∞ conjugate to a concrete representation $L(\Omega, m)$ of X as a space of integrable real functions. *R. E. Fullerton* (College Park, Md.)

2899:

Pae, Mi-Soo. Note on weak topology of Banach space. Kyungpook Math. J. **1** (1958), 65–68.

There is considered the relation of weak compactness in the conjugate space of a Banach space, and the existence of ε-nets. *R. Arens* (Los Angeles, Calif.)

2900:

Phelps, R. R. Convex sets and nearest points. II. Proc. Amer. Math. Soc. **9** (1958), 867–873.

[Part I: same Proc. **8** (1957), 790–797; MR **19**, 432.] Let S be a subset of a normed linear space E. For a given point $z \in S$ denote by S_z the set of all points in E having z as foot or nearest point on S. The set S_z is closed and is, according to part I, convex in any inner product space. It is shown here that any closed convex set in a complete inner product space is a set S_z for a suitable S and $z \in S$; and if every closed convex set in a normed linear space of dimension ≥ 3 is an S_z, then the space is a complete inner product space.

For a set A in a normed linear space E let $K(A)$ be the convex closure of A and $C(A)$ the set of all those points x in E for which no $y \in E$ exists with $\|y-a\| < \|x-a\|$ for all $a \in A$. The property $C(A) \subset K(A)$ characterizes the complete inner product spaces of dimension ≥ 3; in fact, then $C(A) = K(A)$. For dimension 2 the relation $C(A) \subset K(A)$ or $C(A) = K(A)$ means strict convexity of the unit circle. *H. Busemann* (Los Angeles, Calif.)

2901:

Alekseev, V. M. Existence of a bounded function of maximal spectral type. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 5, 13–15. (Russian)

Let $L_\mu^2(\Omega)$ be the Hilbert space [scalar product $(\cdot, \cdot)$] of the square μ-integrable functions defined on an abstract space Ω (μ being a positive measure on Ω). Let $\{E(M)\}$ be a spectral measure in $L_\mu^2(\Omega)$, where M is a Borel set on the real axis. Suppose that $L_\mu^2(\Omega)$ is separable. Then, for all $f \in L_\mu^2(\Omega)$ and $\varepsilon > 0$, there is a bounded $g \in L_\mu^2(\Omega)$ such that the measure $(E(\cdot)f, f)$ is absolutely continuous with respect to $(E(\cdot)g, g)$ and $\|f-g\| < \varepsilon$.

{Reviewer's remark. The second question raised by the author at the end of the note has a negative answer.} *C. Foiaş* (Bucharest)

2902a:

Louhivaara, Ilppo Simo. Zur Theorie der Unterräume in linearen Räumen mit indefiniter Metrik. Ann. Acad. Sci. Fenn. Ser. A. I. no. 242 (1957), 7 pp.

2902b:

Louhivaara, Ilppo Simo. Zur Theorie der Unterräume in linearen Räumen mit indefiniter Metrik. Ann. Acad. Sci. Fenn. Ser. A. I. no. 252 (1958), 11 pp.

The Nevanlinna normal problem is this: Let R be a space with an indefinite Hermitian inner product Q, and a definite Hermitian inner product H, where $|Q(x,x)| \leq H(x,x)$. Let U be an H-closed subspace of R, a an arbitrary element of R. Find the set $A = \{p \mid p \in U, Q(a-p, u) = 0, \text{ all } u \in U\}$. A necessary and sufficient condition for A to be nonempty is that $\int_{-1}^{1} \lambda^{-2} d_\lambda H(E_\lambda a_U^*, a_U^*) < \infty$, where a_U^* is the unique element in U for which $Q(a, u) = H(a_U^*, u)$, all $u \in U$, and $\{E_\lambda\}$ is the resolution of the identity for the bounded (norm 1) self-adjoint operator T defined by $Q(u, v) = H(Tu, v)$, all $u, v \in U$.

The author remarks in the second paper that it is related to one by F. E. Browder [#2819 above] dealing with the Dirichlet problem for nonelliptic self-adjoint partial differential operators. Finally, the author states that the first paper contains an error which invalidates his conclusions in that article. The second work rectifies matters. *B. R. Gelbaum* (Minneapolis, Minn.)

2903:

Banaschewski, Bernhard. Über einen Approximationssatz auf nicht-kompakten Räumen. Arch. Math. **10** (1959), 31–33.

New proof of the reviewer's version of the Stone-Weierstrass theorem [Duke Math. J. **14** (1947), 419–427; MR **9**, 95]. *E. Hewitt* (Seattle, Wash.)

2904:

Horne, J. G., Jr. On primary ideals in $C(X)$. Proc. Amer. Math. Soc. **10** (1959), 158–163.

In the ring $C(X)$ of real-valued continuous functions on a topological space X, an ideal I is defined to be: (1) primary if it is contained in at most one maximal ideal; (2) prime-like if, whenever f, k, $e \in C(X)$ satisfy $f \notin I$, $fe=f$, and $ke \in I$, then $k \in I$; (3) an O-ideal if, whenever $f_1, f_2 \in I$, there exists $e \in I$ (depending on f_1 and f_2) such that $f_i e = f_i$ $(i=1, 2)$. Each ideal J contains an associated O-ideal $L(J)=\{f \in C(X): fe=f \text{ for some } e \in J\}$. The fixed maximal ideals of $C(X)$ are those of the form $M(x)=\{f \in C(X): f(x)=0\}$, for $x \in X$. Theorem 1: For an ideal $P \subset C(X)$, the following conditions are equivalent: (1) P contains a maximal O-ideal; (2) P is a primary ideal; (3) P is a prime-like ideal. Theorem 2: Suppose X is a completely regular space each of whose points is a G_δ-point, and M is a maximal ideal in $C(X)$. Then the following conditions are equivalent: (1) M is a fixed ideal; (2) M contains a principal primary ideal; (3) $L(M)$ is contained in a (proper) principal ideal; (4) M is an isolated point in the space $\mathfrak{M}$ of maximal ideals of $C(X)$, with the sets $\{M \in \mathfrak{M}: f \in M\}$, for all $f \in C(X)$, taken as a base for the open sets; (5) M is the union of principal primary ideals.

The ideal $\bigcap\{[h]: [h] \supset L(M(x_0))\}$, where $[h]$ denotes the principal ideal generated by h, is studied briefly. It is also shown that a maximal ideal M of $C(X)$ is bounded by f in the sense of Pursell [Pacific J. Math. **5** (1955), 963–969; MR **18**, 714] if and only if $[f]$ is a primary ideal contained in M. *C. W. Kohls* (Urbana, Ill.)

2905:

Fujiwara, Kaichirô. Notes sur les demigroupes topologiques des fonctions continues. I, II. Math. J. Okayama Univ. **6** (1956), 71–76; **7** (1957), 185–189.

Let E be a Hausdorff space and $C(E)$ be the set of all continuous functions on E with values in G, where G is a topological semigroup. It is assumed that G has a zero element and a unit, and that if $\{g_\alpha\}$ is a directed set in G then $\lim_\alpha g_\alpha x = x$ holds for an $x \in G$ if and only if $x=0$ or $\lim_\alpha g_\alpha = 1$. $C(E)$ is a topological semigroup under multiplication in the compact-open topology. Let $C_0(E)$ be any sub-semigroup of $C(E)$ containing all the constant functions. A subset S of $C_0(E)$ is called right bounded if for each neighborhood U of 0 there is a neighborhood V of 0 such that $SV \subset U$. A closed ideal I of $C_0(E)$ is called a Ud-ideal if there exists a right bounded directed set $\{e_\alpha\}$ in I such that $\lim_\alpha e_\alpha f = f$ for each $f \in I$.

In part I the Ud-ideals of C_0 are determined as the sets of functions in $C_0(E)$ annihilating appropriate subsets of E. Under further conditions, the maximal Ud-ideals in the Stone topology form a space homeomorphic to E.

In part II, G is the reals, the non-negative reals or the complex numbers. Let E_k be completely regular and $C_0(E_k)$ be a topological semi-group as described above satisfying enough conditions so that $C_0(E_k)$ determines E_k, $k=1, 2$. Suppose that there exists a semi-group isomorphism σ of $C_0(E_1)$ onto $C_0(E_2)$. This induces a homeomorphism τ of E_1 onto E_2. The relations between σ and τ are investigated. In case E is compact and G is the reals these matters were investigated by Milgram [Duke Math. J. **16** (1949), 377–383; MR **10**, 612] and by Yood [ibid. **22** (1955), 383–392; MR **16**, 1125]. *B. Yood* (Eugene, Ore.)

2906:

Parasyuk, O. S. Analytic continuation of generalized functions. Rev. Math. Pures Appl. **1** (1956), no. 3, 51–57. (Russian)

A temperate distribution has an analytic continuation in the upper half plane if and only if its Fourier transform vanishes on the left half-axis. This is a special case of results of Schwartz and of Lions [L. Schwartz, Medd. Lunds Univ. Mat. Sem. Tome Supplémentaire (1952), 196–206; MR **14**, 639]. *W. F. Donoghue, Jr.* (Lawrence, Kans.)

2907:

Ding, Shia-shi. Some properties of a class of Banach spaces. Sci. Record (N.S.) **2** (1958), 59–63.

The author proves an embedding theorem similar to one given by him earlier [Sci. Record (N.S.) **1** (1957), 315–318; MR **20** #6653]. He also establishes the completeness of certain spaces of functions. *G. G. Lorentz* (Syracuse, N.Y.)

2908:

Dinculeanu, Nicolae. Sur la représentation intégrale des certaines opérations linéaires. II. Compositio Math. **14**, 1–22 (1959).

[For part I, see C. R. Acad. Sci. Paris **245** (1957), 1203–1205; MR **19**, 870.]

Let E, F be B-spaces of denumberable type and let T be a locally compact space. Representation in the form $m(x)=\int \langle x(t), x_m'(t)\rangle \mu(dt)$ is given for certain linear functionals m on the space of continuous mappings from T to E with compact support. For linear continuous operators m mapping a space of E-valued integrable functions into F, a representation of the form $m(x)=\int U_m(t)x(t)\mu(dt)$ is given, where U_m is a map from T into the space of continuous linear maps from E to F. If $E=F=A$ where A is a B-algebra with unit, then U_m has its values in A if and only if $m(ax)=am(x)$ for every a in A and every E-valued integrable function x on T. Other more general representation theorems are given. *N. Dunford* (Brooklyn, N.Y.)

2909:

Dinculeanu, Nicolae. Sur la représentation intégrale des certaines opérations linéaires. III. Proc. Amer. Math. Soc. **10** (1959), 59–68.

Let $\mathcal{B}$ be the family of Borel sets in the compact space T, $\mathcal{C}_E(T)$ the space of continuous maps from T to the B-space E, $\mathcal{L}(E, F)$ the space of continuous linear maps from E to the B-space F. Countably additive vector measures on $\mathcal{B}$ to $\mathcal{L}(E, F)$ are considered and it is shown that the regular measures are uniquely determined by the integral of functions in $\mathcal{C}_E(T)$. If E, F are of denumerable

type and F is the dual of another B-space then a regular $\mathscr{L}(E, F)$-valued measure may be written in the form $m(A)=\int_A U(t)\tilde{\mu}(dt)$, where $\tilde{\mu}$ is the variation of m, and U is a mapping of T into $\mathscr{L}(E, F)$. There is a 1-1 correspondence between the regular measures and the linear maps of $\mathscr{C}_E(T)$ into F which are majorized by a positive measure. If $E=F$ is a B-algebra with unit then the measure corresponding to such a majorized map u has its values in E if and only if $u(ax)=au(x)$ for all a in E and x in $\mathscr{C}_E(T)$. The results are proved in the more general case where, instead of one B-space E, a family $E(t)$, $t\in T$, is considered, and instead of a single E-valued integrable function on T, a vector field $x(t)\in E(t)$, $t\in T$, is used. *N. Dunford* (Brooklyn, N.Y.)

2910:

Deprit, André. Contribution à l'étude de l'algèbre des applications linéaires continues d'un espace localement convexe séparé: Théorie de Riesz—théorie spectrale. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **31** (1959), no. 2, 170 pp.

The author makes a thorough study of linear mappings of a vector space which are "close to the identity" in various senses, both from the algebraic and the topological standpoints (the latter when the space is locally convex): for instance, he considers mappings having kernels of finite dimension, or images of finite codimension, and which at the same time are open homomorphisms on their images. He also studies mappings for which the sequences of the iterated kernels or of the iterated images are stationary. Finally, he extends most of the definitions and elementary theorems of classical spectral theory in Banach spaces to more general locally convex spaces. All the author's results are refinements or slight extensions of known ones, and the methods of proof are standard ones. *J. Dieudonné* (Paris)

2911:

Krabbe, G. L. Vaguely normal operators on a Banach space. Arch. Rational Mech. Anal. **3** (1959), 51–59.

The author considers operators in the complex Banach space $l^p=L^p(Z)$, where $p\geq 1$, $Z=\{0, \pm 1, \pm 2, \cdots\}$. For $p<\infty$, $p'=p/(p-1)$, the set $\mathscr{R}=\mathscr{R}(l^p)$ is defined as follows: $T\in\mathscr{R}$ if T is an operator in l^p and also in $l^{p'}$ such that $(Tx|y)=(x|Ty)$ for $x\in l^p$, $y\in l^{p'}$, where $(x, y)\to(x|y)$ is the natural pairing. Any operator of the form T_1+iT_2 with $T_1, T_2\in\mathscr{R}$ is, qua operator in l^p, "vaguely normal". The author finds that the weakly closed algebra of operators in l^p generated by the Hilbert transform $(x_\nu)\to(\sum_{\nu=-\infty}^{\infty} x_\nu(n-\nu)^{-1})$ consists of vaguely normal convolution operators [for which see earlier papers, notably Amer. J. Math. **78** (1956), 42–50; MR **19**, 294], which when $p\geq 2$ have empty residual spectrum {incorrectly called "resolvent set" in this paper}. A spectral theorem is proved. *D. A. Edwards* (Newcastle-upon-Tyne)

2912:

Krabbe, G. L. Convolution operators that satisfy the spectral theorem. Math. Z. **70** (1958/59), 446–462.

Sufficient conditions on a bounded linear operator T in a complex B-space X are given in order that (a) there is an additive operator-valued function E defined on the set of rectangles in the plane which is not necessarily of bounded variation but for which the set of all complex numbers λ for which $E(\{\lambda\})\neq 0$ is contained in the point spectrum of T; (b) the function E exists and $T=\int\lambda E(d\lambda)$ in the weak topology; and (c) the function E exists and is a weakly continuous function of rectangles, T has no residual spectrum, and the set of λ with $E(\{\lambda\})\neq 0$ coincides with the point spectrum of T. Certain convolution operators in l_p have these spectral properties but are not spectral operators. *N. Dunford* (Brooklyn, N.Y.)

2913:

Lyance, V. È. Rings of unbounded operators based on a resolution of the identity and their representations. Dokl. Akad. Nauk SSSR **121** (1958), 801–804. (Russian)

The author considers spectral measures P in a Hilbert space H [see N. Dunford, Pacific J. Math. **4** (1954), 321–354; MR **16**, 142; quoted below as D; and W. G. Bade, ibid., 373–392; MR **16**, 143; quoted as B]. Based on weakening the definition of commutativity of an operator with a spectral measure (by restricting the validity of (2), p. 373 of B), two sets of results are stated by the author: (1) The set $A(P)$ of bounded and densely defined unbounded operators restrictedly commuting with P is a ring complete relative to a suitably defined uniform and strong topology. If P has finite multiplicity s (i.e., H is generated by P acting on a finite subset of s elements of H) then H is isomorphic with a suitable concrete space $L_2(\sigma)$ of s-tuples of functions of a complex variable, square integrable relative to a matrix valued measure σ; this isomorphism carries P into a corresponding Q defined on $L_2(\sigma)$ and $A(P)$ into the corresponding $A(Q)$; this ring is characterized in terms of certain $s\times s$ matrices of functions. (2) By applying his restricted commutativity, the author can state an analog of Theorem 8 of D characterizing unbounded spectral operators A in terms of the generalized nilpotency of certain operators derived from the radical part N of A (see Theorem 8 of D for notation). *G. K. Kalisch* (Minneapolis, Minn.)

2914:

Foguel, S. R. Sums and products of commuting spectral operators. Ark. Mat. **3** (1958), 449–461.

Let F_i, $i=1, \cdots, n$, be disjoint projections whose sum is I and which commute with the scalar operator S, and let E be the resolution of the identity for S. Then the operators $S+\sum_{i=1}^n \mu_i F_i$ and $S(\sum \mu_i F_i)$ are scalar type operators whose resolutions of the identity are given by the expressions $\sum E(\alpha-\mu_i)F_i$, $\sum E(\alpha/\mu_i)F_i$, where α is an arbitrary Borel set in the plane. The following perturbation theorem of the Rellich type is proved. If $\{S_n\}$ is a sequence of commuting scalar type operators in a weakly complete space which converges uniformly to an operator S and for which the Boolean algebra generated by the resolutions of the identity E_n for S_n, $n\geq 1$, is bounded, then S is a scalar operator whose resolution of the identity E has the property that $E(\alpha)x=\lim_n E_n(\alpha)x$ for every Borel set α and every x for which $E(\text{boundary } \alpha)x=0$. These two results are used to show that if S_1, S_2 are commuting scalar type operators in a weakly complete space with resolutions of the identity E, F, and if the Boolean algebra generated by E and F is bounded, then the resolution of the identity of the scalar operator S_1+S_2 [S_1S_2] is given by the expression $G_1(\alpha)x=$

$\int E(\alpha-\mu)F(d\mu)x$ [$G_2(\alpha)x=\int E(\alpha/\mu)F(d\mu)x$] for every Borel set α and every x with $G_1(\text{boundary }\alpha)x=0$ [$G_2(\text{boundary }\alpha)x=0$]. The first results mentioned above are also used to obtain some precise information on the order of the poles of the sum and product of two commuting (not necessarily spectral) operators. These results are then applied to obtain the solution of a problem raised by Hille. The result is as follows. Let C be a linear operator in the complex B-space $\mathfrak{X}$ and let T be the operator in the space $B(\mathfrak{X})$ of bounded linear maps in $\mathfrak{X}$ defined for each A in $B(\mathfrak{X})$ by the equation $T(A)=CA-AC$. Suppose that $\sigma(C)=\{\lambda_1, \cdots, \lambda_n\}\cup\sigma_1=\{\mu_1, \cdots, \mu_n\}\cup\sigma_2$, with $\lambda_i-\mu_i=\delta$, $1\leq i\leq n$, that $\delta\notin(\sigma_1-\sigma(C))\cup(\sigma(C)-\sigma_2)$, and that λ_i, μ_i are poles of order n_i, m_i, respectively, of C; then δ is a pole of the operator T of order $\max_i(m_i+n_i-1)$.

N. Dunford (New Haven, Conn.)

2915:

Fišman, K. M. On the relationship between the method of adjacent systems in special linear topological spaces and certain problems of the theory of perturbation of linear operators in Banach spaces. Dokl. Akad. Nauk SSSR **122** (1958), 22–25. (Russian)

For a bounded operator, $A\in\phi$ means that the range $R(A)$ is closed and its nullity and the deficiency of $R(A)$ are finite numbers. This concept, introduced in Banach spaces, evidently makes sense also for topological linear spaces. The author uses topological linear spaces of a special type, $\mathfrak{C}_r=\prod_{r'<r}\mathfrak{B}_{r'}$, that are limits of non-increasing Banach spaces $\mathfrak{B}_r$, such that $\|f\|_r$ is an increasing function of r and $\lim f_n=f$ means $\lim\|f_n-f\|_{r'}=0$ for all $r'<r$. If A is an operator from $\mathfrak{B}_{r'}$ to $\mathfrak{B}_{\tilde{r}'}$ ($\mathfrak{B}_{\tilde{r}'}$ another such system of Banach spaces) then an operator $\hat{A}$ is induced from $\mathfrak{C}_r$ to $\tilde{\mathfrak{C}}_{\tilde{r}}$. He shows that, if $A\in\phi$, then $\hat{A}\in\phi$. The usual type of perturbation theorem is derived, that asserts that if $\hat{B}$ is sufficiently small, and $\hat{A}\in\phi$, then also $\hat{A}+\hat{B}\in\phi$ and $\kappa(\hat{A}+\hat{B})=\kappa(\hat{A})$ [Gohberg and Kreĭn, Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43–118; MR **20** #3459]. These theorems are then applied to deduce certain results of Evgrafov [Trudy Moskov. Mat. Obšč. **5** (1956), 89–201; MR **18**, 472]. *František Wolf* (Berkeley, Calif.)

2916:

Mascart, Henri. Sur la résolution de certaines égalités entre opérateurs linéaires différentiels. C. R. Acad. Sci. Paris **248** (1959), 906–909.

Let D denote the operation of differentiation and let Q be a fixed linear operator between two function spaces. The author considers the operator equation $D^\beta L=LQ$, where β is a fixed positive integer and L is an unknown operator. Solutions of this equation are investigated, and for particular operators Q certain solutions are determined.

R. E. Fullerton (College Park, Md.)

2917:

deLeeuw, Karel; and Glicksberg, Irving. Almost periodic compactifications. Bull. Amer. Math. Soc. **65** (1959), 134–139.

Let B be a Banach space and S a uniformly bounded semigroup of operators on B. S is called almost periodic (a.p.) [weakly almost periodic (w.a.p.)] if each orbit is conditionally compact in the norm topology [weak operator topology] of B. In both cases, the closure $\bar{S}$ of S in the algebra of bounded operators on B supplied with the weak operator topology is a compact semigroup. The authors announce some results, without proofs, in which such a "compactification" plays a role.

Firstly, results of Jacobs [Math. Z. **64** (1956), 298–338; **67** (1957), 83–92; MR **17**, 988; **19**, 295] are generalized.

Secondly, Eberlein's definition of w.a.p. functions [Trans. Amer. Math. Soc. **67** (1949), 217–240; MR **12**, 112] is extended to topological semigroups, and a compactification theorem (analogous to the theorem of Bohr compactification) is announced. By the compactification the w.a.p. functions on a topological semigroup correspond to the continuous functions on a compact semigroup. For certain semigroups it is possible to define mean values of the w.a.p. functions, put up a Fourier theory, and prove a result which generalizes Eberlein's decomposition [Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 209–219, Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951; MR **13**, 435] of a w.a.p. function into a sum of an a.p. function and a function z with $\mathcal{M}\{|z|^2\}=0$.

Thirdly, a.p. functions on topological semigroups are defined in accordance with the above definition of an a.p. semigroup of operators. (This is weaker than Maak's definition of a.p. functions on semigroups.) For commutative semigroups the authors consider the question of the validity of the approximation theorem, i.e., whether every a.p. function can be uniformly approximated by finite linear combinations of semicharacters, a semicharacter being a complex-valued function χ on the semigroup, satisfying $\chi(xy)=\chi(x)\chi(y)$, $|\chi(x)|\leq 1$. The theorem does not hold in general, but can be established for certain examples of topological abelian semigroups.

E. Følner (Copenhagen)

2918:

Aubert, K. E. Convex ideals in ordered group algebras and the uniqueness of the Haar measure. Math. Scand. **6** (1958), 181–188.

Let G be a locally compact abelian group. Let $L_R{}^1$ be the convolution algebra of real integrable functions on G. Call an ideal A "convex" if $f, g\in A$ and $f\leq h\leq g$ (a.e.) imply $h\in A$. Theorem: The only regular convex maximal ideal in $L_R{}^1$ is $M_R{}^0=\{f: \int f(g)dg=0\}$. And an intersection A of maximal ideals is convex if and only if $A\subseteq M_R{}^0$. (Thus convex ideals are scarce in $L_R{}^1$, in contrast with $C(X)$ where, e.g., every maximal ideal is convex.) *H. Mirkil* (Hanover, N.H.)

2919:

Urbanik, K. Poisson distributions on compact Abelian topological groups. Colloq. Math. **6** (1958), 13–24.

The author calls a probability distribution μ on a compact abelian group G a composed Poisson distribution if there is a finite regular Borel measure ν on G such that

$$\mu = e^{-\nu(G)}\sum_{k=0}^{\infty}(\nu^k/k!),$$

where ν^k is the k-fold convolution of ν with itself. He characterizes such μ by the following: there is a sequence $\{\mu_n\}$ of probability distributions on G with $(\mu_n)^n=\mu$ and $\lim_{n\to\infty}\mu(e)=1$, where e is the identity element of G. A somewhat stronger result is obtained for the case that ν is a point mass. *K. deLeeuw* (Stanford, Calif.)

2920:

Želobenko, D. P. A description of a certain class of Lorentz group representations. Dokl. Akad. Nauk SSSR **121** (1958), 586–589. (Russian)

Let G denote the group of all 2×2 matrices with complex elements and unit determinant. Let D denote the subgroup of all diagonal matrices. If $t\to A_t$ is a representation of the real line by operators in a reflexive Banach space and m is an integer, let $L^{A,m}$ denote the representation of D which takes the matrix whose first diagonal element is $r\exp(i\theta)$ into $\exp(im\theta)A_{\log r}$. By a well-known construction this representation of D induces a corresponding Banach space representation of G which we shall denote by $\tilde{L}^{A,m}$. A few years ago Naĭmark [same Dokl. **97** (1954), 969–972; MR **16**, 218] showed that every reflexive infinite-dimensional Banach space representation of G which is "completely irreducible" is equivalent in a certain weak sense to a representation of the form $\tilde{L}^{A,m}$ where A is one-dimensional. The principal assertion of the present note is that every reflexive Banach space representation of G which does not contain irreducible subrepresentations of a certain kind is a discrete direct sum of representations of the form $\tilde{L}^{A,m}$. The proof is sketched and there is a preliminary result on the structure of a certain group algebra. {The reviewer has taken the liberty of altering the notation and the description of the representations $\tilde{L}^{A,m}$.} *G. W. Mackey* (Cambridge, Mass.)

2921:

Stinespring, W. Forrest. Integrability of Fourier transforms for unimodular Lie groups. Duke Math. J. **26** (1959), 123–131.

Let G be a unimodular Lie group and $\mathscr{L}$ the von Neumann algebra generated by the left regular representation L of G. Let $\Gamma=(L_2(G), \mathscr{L}, m)$ be the dual gage space in the sense of Segal [Ann. of Math. (2) **52** (1950), 272–292; MR **12**, 157], in which integration of operators is defined [Segal, ibid. **57** (1953), 401–457; MR **14**, 991]. This paper is concerned with functions having integrable Fourier transforms for G, where the Fourier transform of a measurable function f on G is meant in an abstract sense as an operator L_f on $L_2(G)$ defined from the convolution, which is studied in the author's paper for a general case [Trans. Amer. Math. Soc. **90** (1959), 15–56; MR **21** #1547]. Denote by C_0^∞ the set of all infinitely differentiable functions on G each with compact support. Let T be an element of the universal enveloping algebra of the right invariant Lie algebra of G satisfying the conditions that T be elliptic as a differential operator and that $(Tf, f)\geq \|f\|_2^2$ for all $f\in C_0^\infty$, and let B be a bounded self-adjoint extension onto $L_2(G)$, commuting with all right translations, of the inverse of the restriction operator of T to C_0^∞. Then Th.1 asserts that: there is a $k\in L_1(G)+L_2(G)$ such that $B=L_k$; and further, for n the dimension of G, if the order of $T>n/2$ [or $>n$], k is in $L_2(G)$ [or continuous, respectively]. Th.3 asserts that: when G is an arbitrary unimodular locally compact group, then for any $f\in L_2(G)$, which is continuous at the unit $e\in G$ and for which L_f is bounded and ≥ 0, one has $L_f\in L_1(\Gamma)$ and $m(L_f)=f(e)$. Th.1 and Th.3 are applied to the integrability of Fourier transforms L_f. Let $X_1, \cdots, X_n$ be a basis for the right invariant Lie algebra and $\Delta=X_1^2+\cdots+X_n^2$. Th.1 is also applied in Th.5 and Th.6 to the effect that, when $T=1-\Delta$ and the operator B is a positive extension, then the function k is ≥ 0 a.e. and in $L_1(G)$. In the final part the author defines the character χ of a subrepresentation U of the representation L and proves two norm (L_1 and L_2) boundedness properties of χ on restricted domains. {Misprints: p. 124, line 29, '$\cap$' should be '$+$'; p. 130, line 22, '$=$' should be '$\leq$'.} *H. Umegaki* (Tokyo)

2922:

Gelbaum, Bernard R. Tensor products of Banach algebras. Canad. J. Math. **11** (1959), 297–310.

Let A and B be commutative Banach algebras, and let C be their tensor product, $A\otimes B$, with the "greatest cross-norm" [R. Schatten, *A theory of cross-spaces*, Princeton Univ. Press, Princeton, N.J., 1950; MR **12**, 186]. If A and B have units, then the space $\Delta(C)$ of complex homomorphisms of C is in a natural way the product $\Delta(A)\times\Delta(B)$. In the present paper, the corresponding proposition for algebras without units is proved. *R. Arens* (Los Angeles, Calif.)

2923:

Leenders, J. H. On a partially ordered matrix-algebra. Simon Stevin **32** (1958), 102–141. (Dutch)

The author gives an extended investigation of the partially ordered set whose elements are all $n\times n$ square matrices over the real field, and $A=(a_{ij})\geq 0$ (the zero matrix) if and only if $a_{ij}\geq 0$ for all values of the indices $i, j=1, 2, \cdots, n$. Section 1 deals with algebraic properties of the matrix algebra, section 2 treats the partially ordered real vector space of dimension n^2 obtained by associating with each matrix (a_{ij}) the vector $A=(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{n1}, \cdots, a_{nn})$, and section 3 studies the systems as a Banach algebra with $\|A\|=\max|a_{ij}|$ $(i, j=1, 2, \cdots, n)$.

L. M. Blumenthal (Columbia, Mo.)

2924:

Yood, Bertram. Homomorphisms on normed algebras. Pacific J. Math. **8** (1958), 373–381.

Let B_1 and B denote real normed Q-algebras and let T be a ring homomorphism or anti-homomorphism of B_1 into B. The author obtains results which assert that, for certain algebras B, T will always be either continuous or closed if the range $T(B_1)$ contains "enough" of B. The main results are the following. (1) If B is the algebra of all bounded operators on a Banach space, $T(B_1)$ contains the operators of finite rank, and $\rho(T(x))\leq\rho(x)$, all $x\in B_1$, where $\rho(x)$ denotes the spectral radius of x, then T is continuous. (2) If B is primitive with minimal ideals, $T(B_1)$ is dense in B, and $T(B_1)\cap I\neq(0)$, for some minimal one-sided ideal I in B, then T is closed and real-homogeneous. (3) If B is strongly semi-simple and $T(B_1)$ is dense in B, then T is closed and real-homogeneous. (4) If B_1 and B are semi-simple, B_1 has dense socle, B has an identity element, $T(B_1)$ is dense in B, and T is real-linear, then T is closed. (5) If B is semi-simple and has dense socle, and if $T(B_1)$ contains the socle of B, then T is closed and real-homogeneous.

As an application, let B be a B^*-algebra with H and K its sets of self-adjoint and skew elements. Then H and K are closed in any normed algebra topology for B.

Another result obtained concerns what the author calls "the spectral extension property" of a normed algebra B.

This property says that the spectral radius of each $x \in B$ is the same in every Banach algebra into which B may be algebraically embedded. The property is necessary and sufficient for a normed algebra to be a Q-algebra in every normed algebra topology. Any 2-sided ideal or any closed subalgebra of the algebra of all bounded operators on a Banach space which contains the operators of finite rank has the spectral extension property.

C. E. Rickart (New Haven, Conn.)

2925:

Pârvu, Monica Pavel. On the resolvent equation. Math. Japon. **5** (1958/59), 35–37.

The author considers the resolvent equation $R(\lambda)-R(\mu)=(\mu-\lambda)R(\lambda)R(\mu)$ for quasi-normed algebras. The latter satisfy the definition of a Banach algebra except that the axiom: $\|\lambda x\|=|\lambda|^r\|x\|$, $0<r\leq 1$, replaces the usual homogeneous axiom ($r=1$). Analogues to the standard theorems of Banach algebras on the structure and expansion of the resolvent are obtained.

E. R. Lorch (New York, N.Y.)

2926:

Daleckiĭ, Yu. L. A linear equation relative to elements of a normed ring. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 165–168. (Russian)

Let $\mathfrak{R}$ be a complex Banach algebra with unit e. For $a \in \mathfrak{R}$, let a_l and a_r be the operators on $\mathfrak{R}$ of left and right multiplication, respectively, by a. Let S_a be the spectrum of $a \in \mathfrak{R}$. Clearly the spectra of a_l and a_r are contained in S_a. Let Γ_a be a smooth contour consisting of a finite number of simple closed arcs, such that Γ_a encloses S_a. Let F_a be the interior of Γ_a plus Γ_a. For a polynomial $P(\lambda,\mu)=\sum_{j,k=0}^{n} c_{jk}\lambda^j\mu^k$ with complex coefficients c_{jk}, and for $a,\ b \in \mathfrak{R}$, form the operator $P(a,b)=\sum_{j,k=0}^{n} c_{jk}a_l^j b_r^k$. We then have

$$(*)\quad P(a,b)=\frac{-1}{4\pi^2}\int_{\Gamma_a}\int_{\Gamma_b} P(\lambda,\mu)(a-\lambda e)_l^{-1}(b-\mu e)_r^{-1}\,d\lambda\,d\mu.$$

By replacing $P(\lambda,\mu)$ in (*) by any function $\phi(\lambda,\mu)$ piecewise analytic in λ and μ for $\lambda \in F_a$ and $\mu \in F_b$, we can define operators $\phi(a,b)$.

Theorem 1. The spectrum of $\phi(a,b)$ is contained in the set of values of $\phi(\lambda,\mu)$.

Theorem 2. The equation (**) $\sum_{j,k=0}^{n} c_{jk}a^j x b^k = y$ has a unique solution $x \in \mathfrak{R}$ for every $y \in \mathfrak{R}$ if the polynomial $P(\lambda,\mu)$ vanishes nowhere for $\lambda \in S_a$, $\mu \in S_b$. This solution can be represented in the form

$$x=-\frac{1}{4\pi^2}\int_{\Gamma_a}\int_{\Gamma_b}\frac{(a-\lambda e)^{-1}y(b-\mu e)^{-1}}{P(\lambda,\mu)}\,d\lambda\,d\mu.$$

A similar expression can be given for $f(\phi(a,b))$ if f is a single-valued analytic function on the set of values of $\phi(\lambda,\mu)$.

In the particular case of (**), $ax-xb=y$, $S_a \cap S_b=0$, $\mathfrak{R}$ is an algebra of operators, theorem 2 becomes a formula of Rosenblum [Duke Math. J. **23** (1956), 263–269; MR 18, 54].

E. Hewitt (Seattle, Wash.)

2927:

Lumer, Gunter; and Rosenblum, Marvin. Linear operator equations. Proc. Amer. Math. Soc. **10** (1959), 32–41.

There is a very extensive overlap between this paper and the paper reviewed immediately above. Both appeared in early 1959 and both generalize quite similarly an earlier paper by Rosenblum [Duke Math. J. **23** (1956), 263–269; MR 18, 54]. To avoid needless repetition of statement, we say merely that this paper is more general than that reviewed above in replacing $P(a,b)=\sum_{j,k=0}^{n} c_{jk}(a_l)^j(b_r)^k$ by $\sum_{j=1}^{n} {}_j a_l\, {}_j b_r$ for $\{{}_j a\}$ an arbitrary set of commuting elements of R and likewise for $\{{}_j b\}$; it is less general, when it considers the special case as above where the ${}_j a$ are powers of a single "a" and the ${}_j b$ of a single "b", in restricting $\phi(\lambda,\mu)$ to be of the form $\sum_{j=1}^{n} f_j(\lambda)g_j(\mu)$ with f_j and g_j separately analytic in appropriate domains. With these changes the main results in the two papers are essentially identical. This paper also contains additional material, for instance, in theorem 1 above giving criteria for $\beta \in \phi(\lambda,\mu)$ actually to be in the spectrum of $\phi(a,b)$.

F. H. Brownell (Seattle, Wash.)

2928:

Tomiyama, Jun. On the projection of norm one in W^*-algebras. II. Tôhoku Math. J. (2) **10** (1958), 204–209.

[For part I, see Proc. Japan Acad. **33** (1957), 608–612; MR **20** #2635.] "Firstly, we obtain the general decomposition theorem of a projection of norm one π from a W^*-algebra M to its C^*-subalgebra N showing that N is decomposed into the maximal W^*-representable direct summand and the rest. Restricting ourself to the case of N being a W^*-representable * subalgebra, we prove that π is decomposed into three parts by three orthogonal central projections z_1, z_2, z_3 of N. The first component is a normal projection of norm one from M to Nz_1, the second singular one to Nz_2 and z_1, z_2 are maximal central projections having these properties. In the last section we discuss on the σ-weak continuity property of π and the relation to the other continuity. We can prove that π is σ-weakly continuous if and only if the kernel of π is σ-weakly closed." (Author's summary)

S. Sherman (Philadelphia, Pa.)

2929:

Altman, M. On involution in Banach spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 33–36.

Let S be the unit sphere in a Banach space E and let $f=I-F$, where F is completely continuous on S to E. Suppose too that $f(x|S) \neq \theta$. Let U be a continuous involution on S to S with $US_1^n \subset S_2^n$ $(n=0,1,\cdots)$, where S_i^n is an n-dimensional sphere. Finally require that $fU(x)/\|fU(x)\| \neq f(x)/\|f(x)\|$, $x \in S$. Then the Leray-Schauder degree of f is odd. This is a slight sharpening of theorems already in the literature due to Krasnosel'skiĭ, Fet and the author [M. A. Krasnosel'skiĭ, Dokl. Akad. Nauk SSSR **73** (1950), 13–15; MR **12**, 111; J. W. Ja vorowski, Bull. Acad. Polon Sci. Cl. III **3** (1955), 289–292; MR **17**, 653; M. Altman, ibid. 409–413; MR **17**, 878]. The proof is given first for the case $f(x)=-fU(x)$ and then this last condition is waived by a homotopy argument.

D. G. Bourgin (Urbana, Ill.)

2930:

Borisovič, Yu. G. On a theorem concerning the critical points of a functional. Mat. Sb. N.S. **42** (84) (1957), 353–360. (Russian)

The subject is that of critical points of an even, positive,

weakly continuous functional, f, on a reflexive Banach space. Furthermore, f is assumed uniformly differentiable in a suitable sphere about the origin and it is supposed $f(x)$ and its gradient vanish at $x=\theta$. The analysis is in terms of the genus of a set [M. A. Krasnosel'skiĭ, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 2 (48), 157–164; MR **14**, 55], a topological invariant resembling category for sets of the sphere S with identification of points corresponding under the antipodal involution. If $[M]_i$ is the class of compact sets M with genus $\geqq i$ on S and if

$$c_i = \sup_{[M]_i} \inf_M f(x),$$

then c_i is a critical value of f on S, and if $c_i = c_{i+1} = \cdots = c_{i+p}$ then $\{x | f(x) = c_i\} \cap S$ contains a compact set of critical points for which the genus exceeds p.

D. G. Bourgin (Urbana, Ill.)

2931:

Veloso, Maria Joana; et Veloso, Eduardo. Sur la notion de dérivée en points frontières dans des espaces localement convexes. Portugal. Math. **17** (1958), 107–122.

An earlier paper by one of the present authors was devoted to a similar topic, but in the more restrictive context of Banach spaces [M. J. Veloso, Univ. Lisboa Revista Fac. Ci. A. (2) **3** (1954/55), 255–278; MR **17**, 511]. Subsequently J. Sebastião e Silva has published work on differential and integral calculus in locally convex topological linear spaces [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 743–750; **21** (1956), 40–46; MR **19**, 561]. The present paper is concerned with extensions of this work of Silva, again with main concern for cases in which a theorem relating to integrals and differentials involves boundary points of the set on which a certain function is differentiable. For the type of differentiability involved, see the review of the cited papers by Silva. *A. E. Taylor* (Los Angeles, Calif.)

2932:

Bartle, Robert G. On the openness and inversion of differentiable mappings. Ann. Acad. Sci. Fenn. Ser. A. I. no. 257 (1958), 8 pp.

The author generalizes some open mapping theorems due to R. and F. Nevanlinna [R. Nevanlinna, same Ann. no. 185 (1955); MR **16**, 806; F. Nevanlinna, ibid. no. 245 (1957); MR **19**, 1067] by exploiting a theorem of Graves [Duke Math. J. **17** (1950), 111–114; MR **11**, 729]. For instance, let f map the ball S_r: $\|x\| < r$ of the real Banach space E_1 into the real Banach space E_2 with $f(\theta_1) = \theta_2$. If the Fréchet derivative of f at x is the linear operator T_x and exists for $x \in S_r$ and

$$\lim_{x \to \theta_1} \|T_x - T_{\theta_1}\| < \mu = \left(\sup_{\|y\|=1} \inf\{\|x\| \,|\, T_{\theta_1} x = y\}\right)^{-1}$$

and if T_θ is 1-1 on E_1 onto E_2, then f maps the ball S_t $(0 < t < r)$ onto an open set G. {Throughout this and antecedent papers in the literature heavy restrictions are placed on the Fréchet derivative, especially at θ. Presumably, the use of differences instead of derivatives would allow weakening of the hypotheses of the existence of T_θ and perhaps of T_x also. Indeed, many other results such as the Hildebrandt-Graves implicit function theorem can possibly be generalized in this way.}

D. G. Bourgin (Urbana, Ill.)

CALCULUS OF VARIATIONS

See also 2821.

2933:

★**Miller, M. Variationsrechnung.** Mathematisch-Naturwissenschaftliche Bibliothek, Bd. 24. B. G. Teubner Verlagsgesellschaft, Leipzig, 1959. iv+133 pp. DM 8.10.

"A text-book for a first introduction to the calculus of variations, requiring a knowledge only of the elements of differential equations. The theoretical explanations are illustrated with many examples worked out in full. The direct methods, e.g. the Ritz method, are also treated with practical purposes in mind." (From the preface.) There are five sections: "die Problemstellung der Variationsrechnung, die Eulersche Differentialgleichung, isoperimetrische Probleme, direkte Methoden, Rand- und Eigenwertprobleme". *S. H. Gould*, (Providence, R.I.)

2934a:

Ciliberto, Carlo. Problemi di Mayer-Lagrange per gli integrali semplici in forma ordinaria. Rend. Sem. Mat. Univ. Padova **28** (1958), 112–152.

2934b:

Ciliberto, Carlo. Alcune osservazioni sui problemi di Mayer-Lagrange per gli integrali semplici in forma ordinaria. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 49–65.

Let C be an absolutely continuous curve $y = y(x)$, $(a \leqq x \leqq b)$, and let $u(x)$ be a solution of the system of differential equations $u_i' = f_i(x, y(x), y'(x), u_1, \cdots, u_n)$, with initial values $u_i(a) = \mu_i$. In the first paper the author shows the existence of a minimum for a function $g(C, \mu) = \Phi(u_1(b), \cdots, u_n(b))$ in suitably restricted classes of curves C and initial values μ. In order to prove the lower semi-continuity of the $u_i(x)$ as functions of (C, μ), it is necessary to assume that each f_i is nondecreasing with respect to each u_k for $k \neq i$, as examples of Graves showed. [See Trans. Amer. Math. Soc. **39** (1936), 456–471, in particular p. 462.] The author makes the notable advance of dropping entirely the requirement of monotonicity of f_i in its argument u_i, and the requirement that f_i be independent of u_k for $k > i$. However, other requirements are added, so that the results do not include the theorems of the cited paper of Graves. In the second paper the author obtains a theorem on lower semi-continuity which apparently leads to existence theorems more general than those of Graves, but not more general than those of the first paper.

L. M. Graves (Chicago, Ill.)

2935:

Pucci, Carlo. Un problema isoperimetrico per la determinazione della forma di una nave. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 31–32.

The author corrects his proof of the existence of the minimum [Atti Accad. Naz. Lincei, Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **4** (1955), 179–218; MR **17**, 1095].

L. Cesari (Lafayette, Ind.)

GEOMETRY

See also 2575, 2607, 2609, 2959.

2936:

Lesieur, L. Théorie algébrique de l'orientation et de la mesure des simplexes. J. Math. Pures Appl. (9) **37** (1958), 245–264.

Let E be an arbitrary set if $n=2$, or the set of "points" of a "geometric lattice" of dimension $n-1$ if $n>2$. An ordered set of n independent points, $(A_1A_2\cdots A_n)$, is called a simplex (when $n=2$ the simplex is called a vector). Let, further, G be a multiplicative group whose identity is 1 and which possesses a central involution which will be represented by -1. Suppose, finally, that there is a mapping into G of all pairs of simplexes whose first $n-1$ points are identical, which mapping fulfils the conditions O_1–O_4 below.

In these conditions the following notation is used: Ω_{n-2} and Ω_{n-1} are ordered sets of $n-2$ and $n-1$ independent points respectively; when $n=2$, Ω_{n-2} may be ignored, and Ω_{n-1} designates a single point; the symbols $(\Omega_{n-1}A_n)$ and $(\Omega_{n-2}AB)$ will represent simplexes, so that the independence of the n points referred to is implied; finally the assumed mapping is designated by the symbolism

$$\frac{(\Omega_{n-1}A_n)}{(\Omega_{n-1}A_n')} = x \in G.$$

The conditions are, then:

$$(O_1) \qquad \frac{(\Omega_{n-1}A_n)}{(\Omega_{n-1}A_n)} = 1;$$

$$(O_2) \qquad \frac{(\Omega_{n-1}A_n)}{(\Omega_{n-1}A_n')} = \frac{(\Omega_{n-1}A_n)}{(\Omega_{n-1}A_n'')}\frac{(\Omega_{n-1}A_n'')}{(\Omega_{n-1}A_n')};$$

$$(O_3) \qquad \frac{(\Omega_{n-2}AC)}{(\Omega_{n-2}AB)}\frac{(\Omega_{n-2}BA)}{(\Omega_{n-2}BC)}\frac{(\Omega_{n-2}CB)}{(\Omega_{n-2}CA)} = -1;$$

$$(O_4) \qquad \frac{(\Omega_{n-1}A)}{(\Omega_{n-1}A')}$$

remains invariant under every permutation of the points of Ω_{n-1}.

The author shows that such a mapping may be extended to one of all pairs of simplexes into G, which mapping has the properties:

$$\frac{\Omega_n}{\Omega_n} = 1; \quad \frac{\Omega_n}{\Omega_n'}\frac{\Omega_n'}{\Omega_n} = 1; \quad \frac{\Omega_n}{\Omega_n'}$$

is an alternating function of the points Ω_n and of the points of Ω_n'; and

$$\frac{\Omega_n}{\Omega_n'} = \frac{\Omega_n}{\Omega_n''}\frac{\Omega_n''}{\Omega_n'}.$$

In other words, such a mapping may be extended to one for which the condition

$$\frac{\Omega_n}{\Omega_n'} = 1$$

is an alternating equivalence relation whose equivalence classes form a representation space for a subgroup of G containing $\{+1, -1\}$.

If this subgroup is $\{+1, -1\}$, we therefore have an orientation defined for the n-simplexes. If G is the multiplicative group of real numbers, $R-\{0\}$, we have a measure of n-simplexes. In the general case, we have a G-measure of n-simplexes. *S. Gorn* (Philadelphia, Pa.)

2937:

Bernhart, Arthur. Polygons of pursuit. Scripta Math. **24** (1959), 23–50.

Exposition and historical account.

2938:

Kooi, O. Inequalities for the triangle. Simon Stevin **32** (1958), 97–101.

If $A_0, A_1, A_2, \cdots, A_n$ are points of a euclidean space, and

$$A_0A_i = \text{dist}\,(A_0, A_i) = \rho \quad (i = 1, 2, \cdots, n),$$

it is easily shown that for every n-tuple of real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$,

$$2\rho^2\cdot(\lambda_1+\lambda_2+\cdots+\lambda_n)^2 \geqq \sum_1^n \lambda_i\lambda_j(A_iA_j)^2.$$

The author deduces from this relation a number of inequalities involving the sides, area, circum-radius and in-radius, of a triangle (to which the names of Weitzenböck, Finsler, Hadwiger, Gerretsen, and Kubota are attached) by special choices of the numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$.
L. M. Blumenthal (Columbia, Mo.)

2939:

Sierpiński, W. Sur les ensembles de points aux distances rationnelles situés sur un cercle. Elem. Math. **14** (1959), 25–27.

Let r^2 be a rational number. Then the author proves that there exists on the circle of radius r a dense set of points so that the distance between any two of them is rational. He further remarks that if on a circle of radius r there are three points whose mutual distances are rational then r^2 is rational.

{Remark by the reviewer: About 15 years ago Ulam asked if there is a dense set in the plane so that the distance between any two of its points is rational. This problem is still unsolved.} *P. Erdős* (Budapest)

2940a:

Fempl, Stanimir. Von einer Kegelnetzlinie. Glas Srpske Akad. Nauka **228** Od. Prirod.-Mat. Nauka (N.S.) **13** (1957), 83–98. (Serbo-Croatian. German summary)

2940b:

Fempl, S. Sur une courbe tracée sur la surface du cône développé. Bull. Acad. Serbe Sci. (N.S.) **20** Cl. Sci. Math.-Nat. Sci. Math. **3** (1957), 47–53.

[The second paper is a French translation of the first.] Analytical method to establish some known properties concerning the development of the circumference of the base of an oblique cone. *F. Şemin* (Istanbul)

2941:

★Гордевский, Д. З. Задачи по аналитической геометрии на образование линий и поверхностей. [Gordevskiĭ, D. Z. Problems in analytic geometry on the

construction of lines and planes.] Izdat. Har'kovsk. Gos. Univ. im A. M. Gor'kil, Kharkov, 1958. 51 pp. 1.40 rubles.

150 exercises, with solutions, in the elementary analytic geometry of curves and surfaces of second order.

2942a:

★Mihailović, D. Vektorska obrada nekih problema analitičke geometrije ravni i prave. [Vector treatment of some problems in the analytic geometry of planes and lines.] Matematička Biblioteka, 5. Univerzitet u Beogradu. Belgrade, 1957. 70 pp.

2942b:

★Mihailović, D. Elementi vektorske algebre i analitičke geometrije u prostoru. [Elements of three-dimensional vector algebra and analytic geometry.] Matematička Biblioteka, 8. Univerzitet u Beogradu. Belgrade, 1958. 179 pp.

Elementary treatments in readable style.

2943:

Bagi, B. B. Equations of associated loci of a conic given in homogeneous co-ordinates. J. Karnatak Univ. **1** (1956), no. 1, 174–175.

The equations of the tangents from a given point to a given conic are presented in a slightly novel form in areal co-ordinates.

2944:

Mandan, Sahib Ram. Hypercones through two quadrics with a common conic in space of four dimensions. Lahore Philos. Soc. J. **8** (1946), 1 p.

The following theorem is proved: given two quadrics q and q' with a common conic p, there are two and only two hypercones through them.

2945:

Di Noi, S. Geometria piana doppiamente parabolica. Period. Mat. (4) **37** (1959), 18–36.

This is an axiomatic treatment of a metric geometry described by the author in Period. Mat. (4) **31** (1953), 296–313 [MR **15**, 461] as "metrica 1-1" (p. 303). This geometry can be briefly described as a study of properties of an affine plane (x, y) invariant under the group of transformations $X = \pm x + a$, $Y = bx \pm y + c$. Distance of two points (x, y), (x', y') is defined as $|x - x'|$. This is not a projective metric in the usual sense as non-coinciding points can have zero distance.

A. Gutwirth (Berkeley, Calif.)

2946:

★Locher-Ernst, Louis. Raum und Gegenraum: Einführung in die neuere Geometrie. Philosophisch-Anthroposophischer Verlag am Goetheanum, Dornach, 1957. 216 pp. 27.50 francs suisses.

This books deals with the projective geometry of three-space, with special emphasis on the duality of point and plane. A fundamental configuration is obtained by four points forming a tetrahedron, and a fifth point not in one of the faces, with which can be constructed a complete hexahedron consisting of 12 points, 12+1 planes and 16+3 straight lines; its polar figure is the complete octahedron. The fundamental structure is the configuration of Reye, which is the self-polar figure containing both complete hexahedron and complete octahedron; it contains twelve complete hexahedra and twelve complete octahedra. Questions of order and division of plane and space by points, lines and planes are studied, again in strict duality, as well as concave regions (Kerne) and concave envelopes of rays (Hüllbereiche) in the plane; with their analogs in space. Five points in space in general position form a complete spatial pentagon; its dual is the complete pentahedron. The first part of the book ("Grundlagen") ends with continuity considerations.

The second part ("Schulung") deals with certain forms of motion obtainable from the traversion of a point range, a pencil of planes or lines ("Dehnen, Umschliessen, Verdichten, Weiten"). The divisions of the plane by four or five elements and the division of space by six planes in general position ("die Sechsheit") are studied, often in considerable detail. Then follow the quadratic saddle surfaces, following the "natural way" from points, straight lines and planes to curved structures; this way also leads to curves and envelopes. This section ends with some remarks of a topological nature, including the Moebius strip and a report of some work by O. Simony on knots.

The third part ("Lehre") deals with harmonic sets, the fundamental theorem of projective geometry, the projective generation of conic sections and the introduction of scales into point ranges, pencils of rays and of planes. Some historical remarks end this rather unusual book; we learn that it has been written as a result of anthroposophical studies into the nature of mathematics, and natural science in general. The term "Gegenraum" is due to Rudolf Steiner (1921), but it seems that Steiner had a more cosmic concept in mind than the pure plane-space, dual to point-space, for which the term is used in this book.

Also those readers who cannot always follow some of the speculations of the author and his circle will find much in this book they can appreciate. There are interesting exercises. The many illustrations are excellent and a pleasure to behold. *D. J. Struik* (Cambridge, Mass.)

2947:

Derry, Douglas. On polygons in real projective n-space. Math. Scand. **6** (1958), 50–66.

The paper deals with the class $\{\pi_n\}$ of those polygons π_n of real projective n-space such that (1) π_n is not contained in a hyperplane, and (2) no hyperplane intersects π_n in more than n points. "The principal result is that each such polygon may be obtained from a polygon connecting the vertices of a simplex by a succession of inscribed polygons" defined in the paper.

L. M. Blumenthal (Columbia, Mo.)

2948:

Pickert, Günter. Vertauschbarkeit der Linksadditionen in einer affinen Ebene. Arch. Math. **9** (1958), 147–151.

In a projective plane, suppose given two distinct points U, V, a line η, distinct from UV, through V, and a point W, distinct from U and V, on UV. The author associates

with each line η_x, distinct from UV, through V, a point-mapping of η, called left-addition of x, namely the result of following a projection from η to η_x with centre W by a projection from η_x to η with centre U. He considers the configuration, call it the (U, V, η)-configuration, which expresses the condition that, for fixed U, V, η and variable W, all the left-additions commute with one another. By giving a more algebraic formulation of left-additions and using them to discuss the existence of collineations, he gives a simple proof of the following: (*) The (U, V, η)-configuration implies the little Desargues' theorem with centre V, axis UV. He points out that this result is a reformulation of one due to Gleason [Amer. J. Math. **78** (1956), 797–807; MR **18**, 593]. The converse of (*) is false; however, the author shows that the little Desargues' theorem with centre V, axis UV, and (for any fixed choice of W on UV, distinct from U, V) the Thomsen configuration in the (U, V, W)-net, together imply the (U, V, η)-configuration. When U, V, η are also allowed to vary, the (U, V, η)-configuration becomes a quadrangle-configuration (VS-Satz) previously studied by the author [G. Pickert, *Projektive Ebenen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 399].

R. H. Bruck (Madison, Wis.)

2949:

★**Pickert, Günter. Die Assoziativität der Multiplikationen und Additionen in einer projektiven Ebene.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 30–40. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

Let O be the origin of a coordinate system and U, V the improper points of the coordinate axes of a plane. Then the associativity of multiplication in this coordinate system is characterized by the Reidemeister condition in the (O, U, V)-net, or, in short, the (O, U, V)-Reidemeister condition. [The notation follows the author's book *Projektive Ebenen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 399.] According to W. Klingenberg [Abh. Math. Sem. Univ. Hamburg **19** (1955), 158–175; MR **16**, 950], either the associativity of all multiplications, or the validity of the (O, U, V)-Reidemeister condition for all noncollinear triples O, U, V, implies the Desargues theorem. The author uses Klingenberg's method for deriving specializations of the Desargues theorem from weaker assumptions. (1) Let U, V be distinct points and $\gamma(\neq UV)$ a line through V. Then the plane is (U, V)-transitive if and only if the (O, U, V)-Reidemeister condition holds for all points O which lie neither on UV nor on γ. (2) Let O and O' be not on UV, and $OV \neq O'V$. Then the plane is (U, V)-transitive if and only if the Desargues (V, UV)-theorem and the (O, U, V)- as well as the (O', U, V)-Reidemeister condition hold. (3) Let O, U, V be a noncollinear triple and U' a point on UV, $U \neq U' \neq V$. Then the plane is (U, V)-transitive if and only if the Desargues (V, UV)-theorem and the (O, U, V)- as well as the (O, U', V)-Reidemeister condition hold.

In the second part of the paper the author deals with A. M. Gleason's theorem [Amer. J. Math. **78** (1956), 797–807; MR **18**, 593] stating that the associativity of all the additions in a finite projective plane of prime power order implies the Desargues theorem. A characterization of the plane is given starting from the group of left additions (J^{ab}_1 in Gleason's notation). The author expresses his hope that this approach might turn problems connected with this subject into problems about permutation groups.

R. Artzy (Haifa)

2950:

Salzmann, Helmut. Topologische projektive Ebenen. Math. Z. **67** (1957), 436–466.

Eine lokalkompakte projektive Ebene besitzt eine abzählbare Basis. Sie braucht nicht kompakt zu sein (im Widerspruch zu einer Behauptung von L. A. Skornyakov [vgl. Trudy Moskovsk. Mat. Obšč. **3** (1954), 347–373; MR **16**, 60]. Sie ist es, sobald man noch Zusammenhang voraussetzt. Dasselbe gilt, wenn die Topologie auf Anordnung beruht; die Anordnung ist dann von selber archimedisch. Auch mit Inzidenzsätzen kann man die Kompaktheit lokalkompakter projektiver Ebenen erzwingen, z.B. wenn man die Existenz eines Punktes p und einer nichtinzidenten Geraden G mit einer Kollineationsschar fordert, die p und G punktweise festläßt und übrigens auf jeder Geraden durch p transitiv ist.—Ist der additive Loop eines Ternärkörpers der lokalkompakten projektiven Ebene eine kommutative Gruppe, so ist er direkte Summe von n Additionsgruppen der reellen Zahlen. Die projektiven Geraden einer zur reellen projektiven Ebene homöomorphen projektiven Ebene sind topologische Kreise.—Eine zur reellen Ebene homöomorphe Translationsebene ist zu ihr isomorph.—Zu jedem topologischen Alternativkörper gibt es eine genau projektive Ebene.

H. Freudenthal (Zbl **78**, 341)

2951:

★**Lehti, Raimo. On the construction of elementary models for finite plane geometries.** Treizième congrès des mathématiciens scandinaves, tenu à Helsinki, 18–23 août 1957, pp. 141–170. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

The author is interested in "concrete" models of finite projective planes in which each point [line] of the finite plane is represented by one or more points [lines] "drawn on real paper", using as few elements as possible for such a representation. Illustrations are given for the geometries over GF(2), GF(3), GF(5) and GF(2^2).

F. A. Behrend (Melbourne)

2952:

Dembowski, Peter. Homomorphismen von λ-Ebenen. Arch. Math. **10** (1959), 46–50.

A λ-plane is a set of points and lines with an incidence relation such that any two points [lines] are incident with exactly λ lines [points]. A homomorphism of a λ-plane onto a λ'-plane is a point-point, line-line mapping preserving incidences. A residue class of such a homomorphism ϕ is the set of all points [lines] mapped by ϕ onto a particular point [line]. The author shows that all residue classes of a homomorphism of a λ-plane onto a projective plane have the same cardinality, and then proves the first of his two theorems: a homomorphism of a λ-plane onto a projective plane with a finite residue class is an isomorphism (i.e., 1-1). Some use is made of earlier work [Math. Z. **69** (1958), 59–89; MR **20** #255]. The second theorem of the paper is: Let E' be a λ'-plane and $\lambda' \leq \lambda$;

then there exists a λ-plane E which has E' as a homomorphic image. In fact, E is the free extension of E' to a λ-plane. *R. Sandler* (Chicago, Ill.)

2953:

★**See, Michele. Sui k_q-archi di indice h.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 133–135. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

A k_q-arc is a set of k points of a plane over a $GF(q)$, such that no three points of the set are collinear. A k_q-arc is said to be complete when there is no $(k+1)_q$-arc containing it; then each point of the plane lies on a (positive) number of chords of the k_q-arc, and the minimum number h of chords issuing from a point of the plane is called the index of the k_q-arc. Certain inequalities connecting the characters q, k and h are given, in connection with the problem of classifying the complete k_q-arcs. *B. Segre* (Rome)

2954:

★**Scafati, Maria. Sui 6-archi completi di un piano lineare $S_{2,8}$.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 128–132. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

With the notation of the previous review, given a 5_8-arc, all the complete 6_8-arcs containing it are constructed. *B. Segre* (Rome)

2955:

Karzel, Helmut. Quadratische Formen von Geometrien der Charakteristik 2. Abh. Math. Sem. Univ. Hamburg **23** (1959), 144–162.

The author continues his investigations of generalized absolute geometries $(\mathfrak{G}, \mathfrak{E})$ [see, in particular, Arch. Math. **6** (1955), 284–295; MR **17**, 776] to obtain a representation of the group $\mathfrak{G}$ for geometries of characteristic 2 by means of ternary quadratic forms. The representations are to subgroups of the ternary orthogonal group with reference to the form over a field of characteristic 2. The technique is to imbed the geometry in a desarguesian plane and to obtain the quadratic form from a consideration of the equations of the reflections which serve as a basis for the definition of the generalized geometries; these equations are expressible in terms of the coordinate field of the desarguesian plane.

For groups $\mathfrak{G}$ with a non-trivial center $\mathfrak{Z}$, a geometry $(\mathfrak{G}^*, \mathfrak{E}^*)$ with $\mathfrak{G}^* = \mathfrak{G}/\mathfrak{Z}$ is first obtained and the quadratic form derived for this geometry. The representation of $\mathfrak{G}$ is then accomplished by a direct product.

Specific discussion is given to singular 'Lotkern' geometries, in which case the form is of rank 2, and to elliptic 'Lotkern' geometries where the rank is 3. As a consequence of the investigations it is possible to characterize the division rings which may coordinatize projective planes in which the general geometries of characteristic 2 are imbeddable. They are commutative and contain an irreducible separable quadratic polynomial. *J. D. Swift* (Los Angeles, Calif.)

2956:

Takasu, Tsurusaburo. A non-linear N. E. geometry with a general p-ic surface as absolute. Yokohama Math. J. **5** (1957), 171–199.

This is a sketch of a projective geometry with an absolute of the third order $a_{ijk}x^ix^jx^k=0$. Natural coordinates are found and the 12-parameter transformation group, with a 6-parametric subgroup. Length and angle can be defined, and a trigonometry developed. This also allows interpretations in terms of sphere geometries. Generalization to absolutes of order p are possible. *D. J. Struik* (Cambridge, Mass.)

2957:

Tamássy, Lajos. Über eine Verallgemeinerung der Möbiusschen Kreisgeometrie. Acta Univ. Debrecen. **2** (1955), 137–143. (Hungarian. German summary)

Diejenigen umkehrbar eindeutigen und inzidenztreuen Abbildungen der Ebene ε werden erörtert, welche die Vereinigungsmenge einer gewissen Menge $\mathfrak{H}_1$ von Hyperbeln und einer Menge $\mathfrak{H}_2$ von Geraden in sich selbst, und die Punkte der Ebene ε in Punkte derselben Ebene überführen. Die Elemente von $\mathfrak{H}_1$ sind gleichseitige Hyperbeln mit parallelen Hauptachsen der—in einer von der projektiven Geometrie abweichenden—durch ideale (unendlichferne) Elemente erweiterten euklidischen Ebene. $\mathfrak{H}_2$ is die Menge der Geraden der euklidischen Ebene mit Ausnahme der Asymptoten der genannten Hyperbeln, ergänzt durch einen einzigen gemeinsamen idealen (unendlichfernen) Punkt. ε ist die euklidische Ebene, ergänzt durch die soeben eingeführten idealen Punkte. Aus dieser Geometrie, bzw. aus der Möbiusschen Kreisgeometrie kann man durch achsenparallele Streckung auch auf andere Kegelschnittscharen gegründete Geometrien erhalten. Unter allen diesen Geometrien ist die Möbiussche durch die Winkeltreue ausgezeichnet.

Aus der Zusammenfassung des Autors

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 2895a-b, 2896, 2900, 2988.

2958:

Tamássy, L. Über Kurvenbögen, die gewisse Extrema liefern. Publ. Math. Debrecen **6** (1959), 90–100.

Let O, A, B be three points on the plane, and let S be the family of arcs with continuous and non-negative curvature κ which unite A with B tangent at the end points respectively to OA and OB and having $\kappa(A)=\kappa(B)=0$. Generalizing a result of Moore and Török [Acta Sci. Math. Szeged **15** (1954), 157–163; MR **15**, 826], the author investigates: (a) the curves of S with least upper bound of κ and (b) those with the additional condition $\Delta\kappa/\Delta s \leqq m$ (m=given constant). If $OA \leqq OB$, and r_0 denotes the radius of the circle passing through A tangent to OA and OB, the least upper bound (not accessible) for all curves of S is $1/r_0$. The solution of (b) is given by a curve with $\kappa = ms$ for $s \leqq \eta/m$ and $\kappa = \eta$ for $s > \eta/m$ from A to the bisector of the angle AOB and then continued by symmetry (the constant η is determined by a transcendental equation which depends on m). *L. A. Santaló* (Buenos Aires)

2959:

Bottema, O. A minimax property for the triangle. Nieuw Tijdschr. Wisk. **46** (1958/59), 213–218. (Dutch)

For each point P in the plane of a triangle ABC, the author considers the number $\lambda(P)=k_1/k_2$, where k_1 is the sum of the squares of the distances from P to the three vertices while k_2 is the sum of the squares of the distances from P to the three sides. The locus of points P for which λ has a given value is a conic belonging to the pencil $k_1-\lambda k_2=0$. The characteristic equation for this pencil is a cubic whose coefficients are certain functions of the sines of the angles of the triangle. The roots are all positive. Calling them λ_1, λ_2, λ_3 in increasing order, he observes that the possible range of values for $\lambda(P)$ is from λ_1 to λ_3. The minimum satisfies $1<\lambda_1\leq 2$, with $\lambda_1=2$ only when the triangle is equilateral (and P is at infinity). The maximum satisfies $\lambda_3\geq 4$, with $\lambda_3=4$ only when the triangle is equilateral (and P is at the center). The conic $k_1-\lambda k_2=0$ is an ellipse when $\lambda_1<\lambda<3/2$ and again when $3<\lambda<\lambda_3$; it is a parabola when $\lambda=3/2$, a hyperbola when $3/2<\lambda<3$, and a pair of parallel lines when $\lambda=3$.

H. S. M. Coxeter (Toronto, Ont.)

2960:

Rogers, C. A.; and Shephard, G. C. Some extremal problems for convex bodies. Mathematika **5** (1958), 93–102.

Let E be any measurable subset of Euclidean n-space, $V(E)$ its volume, $C(E)$ its least convex cover. If H, K are convex, define $V^*(H,K)$ to be

$$\sup[V(C(H\cup(K+x)))|x\in H-K].$$

The least values, for convex bodies H, K of given volume, are found for the functions $V^*(H,K)$, $V^*(K,-K)$, $V^*(K,K)$. The greatest value of the last function is also found, the greatest values of the other two being already known. The authors deduce that "no convex body has the property that, for each direction, it can be well approximated by a cylinder".

The proof depends on Steiner symmetrization, and also on the new idea of linear parameter systems $C(t)$ of convex sets. For such systems, $V(C(t))$ is shown to be a convex function of t.

A. M. Macbeath (Dundee)

2961:

Fast, G. [Fast, H.] Area of a generalized circle as a function of its radius. I, II. Fund. Math. **46** (1959), 137–163. (Russian)

Given a plane set M, the generalized circle C of centre M and radius r is the set of the points of the plane which are distant at most r from M. The function $\phi(r)$ will denote the amount by which the area of C exceeds that of an ordinary circle with the same radius. This function was introduced by Minkowski, who observed that $\phi(r)$ is linear in the case in which M is convex. The author considers the more general case in which M is connected and proves that $\phi(r)$ is then convex: more precisely, $\phi(r)$ is then linear in a certain interval $a<r<b$ (possibly empty) and is strictly convex for $r<a$ and for $r>b$. Here a is defined as the supremum of the radii of ordinary circles contained in bounded components of the exterior of M; the quantity b (which may be $\leq a$) is defined, in a more complicated way, in terms of the simply-connected completion of M.

L. C. Young (Madison, Wis.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 2684, 2903, 2929, 2939.

2962:

★Čech, Eduard. Topologické prostory. [Topological spaces.] With supplements: J. Novák, Construction of certain important topological spaces; M. Katětov, Fully normal spaces. Nakladatelství Československé Akademie Věd, Prague, 1959. 524 pp. 38 Kčs.

Here are the headings of paragraphs along with the number of corresponding subparagraphs. § 1. Sets. Algebra of sets. Transformations (pp. 13–22; 6 subparagraphs). § 2. Countable sets. Cardinalities (pp. 23–29; 4). § 3. Ordered sets (pp. 30–56; 10). § 4. Topological spaces and F-spaces (pp. 57–95; 13) (a topological space is any ordered pair (P,u) of a set P and a mapping u of the set 2^P of parts of P into itself such that $u\emptyset=\emptyset$ if X is an empty or one-point set, and satisfying the distributive law: $u(X\cup Y)=uX\cup uY$; F-space means any topological space in which the closure is closed). § 5. Separation axioms (pp. 102–112; 5). § 6. Examples of topological spaces (pp. 113–141; 5) (ordered spaces, cartesian multiplication, limit spaces, etc., are included). § 7. Continuous functions (pp. 142–169; 5). § 8. Coverings (pp. 170–213; 5). § 9. Metric spaces (pp. 214–264; 6). § 10. Connected sets (pp. 265–318; 8) (§ 10.6, pp. 294–309, is dedicated to cyclic orders and cyclic spaces). § 11. Locally connected spaces (pp. 319–342; 4). § 12. Some more recent results (pp. 343–382; 4). § 12 contains a study of FH-closed spaces. An FH-space, i.e. a Hausdorff space, is FH-closed if and only if every open cover contains a finite number of members the closures of which cover the space (§ 12.1.2). A continuous FH-image of an FH-closed space is again FH-closed (§ 12.1.8). The existence of FH-closed closures of FH-spaces is proved; i.e., for every FH-space P there exists an FH-closed space R such that (1) P is an everywhere dense set in R, (2) if f is a continuous function from P to some FH-space Q, then there exists a set M between P and R and a continuous function g from M to $\overline{f^1M}\subseteq Q$ such that $f=g$ in P.

An important part of § 12 is dedicated to the study of characters and β-closures of topological spaces, proving Pospišil's theorem of existence of a Hausdorff space of cardinality 2^{2^x} having an everywhere dense set of cardinality x. In particular, according to § 12.3.2, for every cardinal $e\geq\aleph_0$ there exists a completely regular not completely normal space of cardinality 2^{2^e} containing an isolated everywhere dense set Q of cardinality e.

Appendix I (by J. Novák), Construction of some important topological spaces (pp. 383–406). § 1. A regular F-space on which every continuous function is constant (the space is an L-space obtained by identification method in the cartesian product $A\times B\times N\times B$, where A, B are infinite sets of cardinalities a, b, with $a<b$; N is the set of natural numbers). In § 2 one considers, in connexion with a space (P,u), the spaces (P,u^ε), where $u^{\varepsilon+1}=uu^\varepsilon$, $u^\lambda X=\bigcup_{\varepsilon<\lambda}u^\varepsilon X$; for any $X\subseteq P$ let $\varphi(X)$ be the first ordinal such that $u^{\varphi(X)}X$ is closed; let $\phi(P,u)=\{\varphi(X)|X\subseteq P\}$; one examines the nature of any set H of ordinals satisfying $H=\phi(P,u)$ for some space (P,u). In § 3 one constructs two regular $\aleph_0$-compact spaces whose combinatorial product is not $\aleph_0$-compact.

Appendix II (by M. Katětov), Fully normal or paracompact spaces (pp. 407–495), consists of a preface and

10 paragraphs. Here are the titles of the first 8 paragraphs: § 1. Locally finite systems. § 2. Locally finite covers. § 3. Pseudo-metrics (like metrics, the axiom $\rho(x, y) = 0 \Leftrightarrow x = y$ being replaced by $\rho(x, x) = 0$). § 4. Normal covers (defined by means of six equivalent propositions; an open cover $\{G_\alpha, \alpha \in A\}$ of a space P is normal, if there exists an ω-sequence of open covers G_n such that G_{n+1} is starlike inscribed in G_n ($n = 0, 1, \cdots$), $G_0 = G$; i.e., if, for each $x \in P$, if $x \in X \in G_{n+1}$ then $X \subseteq Y$ for some $Y \in G_n$, whatever be $n < \omega$). § 5. Fully normal spaces. § 6. Hereditary fully normal spaces. § 7. Local properties. § 8. Countable fully normal spaces.

The appendix, with numerous examples, is a quite complete study of paracompact spaces (by the way, uniform structures, uniform spaces and proximity spaces also are mentioned).

The book is dedicated to the memory of B. Pospišil (1912–1944) and on pp. 507–516 contains an analysis of his scientific work. An index, pp. 517–524, indicating not only the page but also the line in which an item is introduced, is the last part of the book.

The book contains very many examples and no drawings at all. There are some special notations and terminology. E.g., "function" means "single-valued function", the inverse of a function f is designated by f_{-1}; f^1 means the induced mapping of the parts of the domain of f. The style of the book is rather condensed.

Đ. Kurepa (Zagreb)

2963:

Császár, Á.; et Czipszer, J. Sur les courbes irramifiées. Acta Math. Acad. Sci. Hungar. **9** (1958), 315–328.

Elementary (and to some extent new) proofs are given of the following known theorems. I. If every point of a nondegenerate, connected, separable, Hausdorff space X is of (Menger) order 2 or less, then X is homeomorphic with a plane simple continuous curve (i.e., an arc, a ray, a simple open curve or a simple closed curve). Furthermore, if X is metric one may drop the separability assumption. {Reviewer's remark: In fact, if X is semi-metric the separability assumption may be omitted.} II. If every point of a connected Hausdorff space X is of (Menger) order 2 or less, then X is a generalized simple continuous curve.

Finally the authors generalize one of Menger's theorems as follows: If X is a nondegenerate, connected, compact (= bicompact) Hausdorff space which contains no point of infinite order and at most a finite number of points of order greater than 2, then X is the union of a finite collection of generalized arcs no one of which contains a non-end point of another. The non-compact case is also considered.

The authors point out the relation of their work to Frankl's [Fund. Math. **11** (1928), 96–104] and the reviewer's [Bull. Amer. Math. Soc. **45** (1939), 623–628; MR **1**, 45]. *F. B. Jones* (Chapel Hill, N.C.)

2964:

Császár, Á. Sur les courbes atriodiques. Acta Math. Acad. Sci. Hungar. **9** (1958), 329–332.

Suppose that the complete Moore space R is nondegenerate, connected and locally connected. Then if R is atriodic, R is homeomorphic to an arc, a ray, a simple open curve or a simple closed curve [F. B. Jones, op. cit., review above; Theorem 16]. The author gives a new elementary proof of this theorem for the metric case. {Reviewer's remarks: Every metric space is a Moore space and every Moore space is semi-metric; the above theorem does not generalize to semi-metric spaces; in this respect it is unlike the main theorem in the preceding paper.} *F. B. Jones* (Chapel Hill, N.C.)

2965:

Armentrout, Steve. Concerning a certain collection of spirals in the plane. Duke Math. J. **26** (1959), 243–250.

"Let Σ be a family such that G belongs to Σ if and only if G is a collection of mutually exclusive arcs in the plane such that there exist a straight line L and a side D of L such that (1) each arc of G has one endpoint on L and lies, except for this point, in D, (2) G^* (the union of the sets of G) is bounded, and (3) if g is an arc in G and A is the endpoint of g not on L, then g spirals down on A. The result of this paper is the following: There exists a collection Γ in Σ such that the point set M, to which P belongs if and only if some arc of Γ spirals down on P, is an arc. [For a definition of a spiral, and of spiralling down, see R. L. Moore, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 207–213; MR **14**, 783]." (Author's introduction)

R. Ellis (Philadelphia, Pa.)

2966:

Isiwata, Takesi. Normality of the product space of a countably compact space with its any compactification. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 181–184.

The paper is an axiomatic treatment of the examples of non-normal spaces and of spaces with unique uniform structure which are usually constructed from spaces of ordinal numbers. *J. Isbell* (Lafayette, Ind.)

2967:

Kenyon, H.; and Morse, A. P. Runs. Pacific J. Math. **8** (1958), 811–824.

The paper gives re-wordings of the usual topological and limit operations by using binary relations R (i.e., subsets of cartesian products). In particular, by translating the fundamental axiom of topology, viz., that any two neighborhoods of any point p contain a neighborhood of p, they define a run as any non-vacuous relation R such that to each $x \in \mathrm{pr}_1 R$, $y \in \mathrm{pr}_1 R$ there exists a $z \in \mathrm{pr}_1 R$ such that $(z, t) \in R \Rightarrow (x, t) \in R$, $(y, t) \in R$. Roughly speaking, this means that for any pair x, y of points there exists a point whose "followers" follow both x and y. In the particular case that R is transitive, the runs coincide with the directions in the sense of Moore-Smith. E.g., if T is a set topologizing a set S, then for every point $p \in S$ the set of ordered pairs (β, x) such that $p \in \beta \in T$, $x \in \beta$ constitutes a run which converges to p in the topology T. Various definitions and notations are introduced. For example, R runs in A if and only if R is a run and the range of R is a part of A; R is eventually [resp. frequently] in A provided R is a run and the set Rx of all the y satisfying $(x, y) \in R$ is such that $Rx \subseteq A$ [resp. $Rx \cap A \neq \emptyset$] for some [each] $x \in \mathrm{pr}_1 R$. S is a subrun of R if both R, S are runs and to each $x \in \mathrm{pr}_1 R$ is associated a $y \in \mathrm{pr}_1 R$ satisfying $Sy \subseteq Rx$. One says that R runs the same as S provided R is a subrun of S and S one of R.

Here are some runs: $R_1=\{(x, y);\ 0\leq x\leq y<\infty\}$; $R_2=\{(x, y);\ 0<x<\omega,\ 0\leq x\leq y<\infty\}$; $R_3=\{(x, y);\ x\leq y;\ x, y$ finite sets$\}$; $R_4=\{(P, \xi);\ P$ is a countable disjoint family of nonvacuous L-measurable sets$\subseteq R[0, 1]$ and $\bigcup P=R[0, 1]$; ξ is any real-valued function that is defined on a refinement P' of P of the same structure as P and satisfying $\xi(x)\in x$ for each $x\in P'\}$. By means of the run R_4 one expresses the Lebesgue integral as the limit of sums analogous to Riemann sums. An adequate translation is given for limits of functions (and of sequences in particular), of topological convergence, of the Tychonoff theorem on the topological product of bicompact spaces, of Zorn's lemma, etc.

The aim of the paper is to indicate that runs are adequate tools for topological convergence and limits; speaking of filters and directions, the authors conclude: "We feel that runs retain the virtues of directions and at the same time remove their inadequacies."

Đ. Kurepa (Zagreb)

2968:

Meyer, Burnett; and Sprinkle, H. D. Two nonseparable complete metric spaces defined on [0, 1]. Pacific J. Math. **8** (1958), 825–828.

If A and B are subsets of $[0, 1]$, let $\rho(A,B)=m^*(A-B)+m^*(B-A)$ and let $\delta(A, B)=m^*[(A-B)\cup(B-A)]$ (m^* denotes the exterior Lebesgue measure). Both ρ and δ are pseudo-metrics for the space of all subsets of $[0, 1]$. By forming the natural equivalence classes (i.e., A is equivalent to B if $\rho(A, B)=0$ or $\delta(A, B)=0$), one obtains (topologically identical) metric spaces. Since there exist uncountably many nonintersecting subsets of $[0, 1]$ each having exterior measure equal to one, both spaces are nonseparable.

These and the other results of this paper are known, being for the most part contained in a paper by M. Fréchet [Calcutta Math. Soc. Bull. **15** (1924), 1–8]. Among other properties of such spaces, Fréchet points out that they are locally arcwise connected.

F. B. Jones (Chapel Hill, N.C.)

2969:

Wang, Shu-tang. On a theorem for the uniform spaces. Sci. Record (N.S.) **2** (1958), 338–342.

The author's aim is to give elegant proofs for a number of theorems obtained by the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A **60** (1957), 421–435; MR **20** #6082; see also RŽMat **1959** #187]. His proofs are based on the following basic theorem. If X is a uniform space with a structure base of cardinality $\leq u$ then X admits a base of cardinality $\leq u$ for its open sets if (and only if) the following proposition holds: Every subset S of cardinality $>u$ has a contact point of order ≥ 2 in X. A point x is said to be a contact point of order $\geq m$ of the set S if for every neighborhood N_x of x the power of $N_x\cap S$ is $\geq m$. In the time between the submission and the publication of the author's article another paper was published on the subject by the reviewer [Pacific J. Math. **8** (1958), 721–734; MR **20** #6083]. Theorem 2.2 of this article contains the basic theorem as a special case. (Set $m=u$.) Similarly, theorem 5′ of the author is the case $m\geq u$ of theorem 4.2 of the reviewer. The author also proves that if X has a structure base of cardinality $\leq u$ and every subset of power $\geq u$ has a contact point of order $\geq u$, then for every covering of X we can select a subcovering of power $<u$ ($u>1$). He also remarks that two theorems in the reviewer's first paper are incorrectly stated and gives correct formulations by replacing "accumulation point" by "contact point of order ≥ 2". {All through the reviewer's first paper it is understood that ξ is an accumulation point of S if every open neighborhood O_ξ of ξ contains at least two points of S. The reviewer tried to point this out in the introduction but at the moment he failed to realize that ξ need not be in S. Everything becomes clear by looking at any of the proofs. In the second paper the reviewer was careful enough not to explain what is meant by a derived set.}

I. S. Gál (New Haven, Conn.)

2970:

Pae, Mi-Soo. Note on a mapping of connected components. Kyungpook Math. J. **1** (1958), 81–83.

Let f be a closed mapping from X to Y. Suppose whenever $f(A)$ and $f(B)$ meet (where $A, B\subset X$) then there is an $x\in A^-\cap B^-$ such that $f(x)\in f(A)\cap f(B)$. If f^{-1} is also closed then it establishes a 1:1 correspondence between the components of A and those of B.

R. Arens (Los Angeles, Calif.)

2971:

Rodnyanskiĭ, A. M. Mappings into a Euclidean space of the product of a topological space and a Euclidean. Mat. Sb. N.S. **46** (88) (1958), 27–60. (Russian)

The author proves the theorems stated in a previous paper [Dokl. Akad. Nauk SSSR **115** (1957), 659–662; MR **20** #1295].

H. Komm (Troy, N.Y.)

2972:

Berstein, I. On the factorizability of maps of S^n into S^n. Fund. Math. **45** (1958), 138–142.

The author answers several questions raised by K. Borsuk by proving the following theorem: There exists no n-dimensional paracompact space X, $n\geq 2$, such that a continuous map φ of an n-sphere S^n into itself is inessential if and only if it is representable as $\varphi=\varphi_2\varphi_1$, where φ_1 and φ_2 are continuous maps of S^n into X and of X into S^n respectively.

C. T. Yang (Philadelphia, Pa.)

2973:

Borsuk, K. Concerning the classification of topological spaces from the stand-point of the theory of retracts. Fund. Math. **46** (1959), 321–330.

Let Σ be a given class of non-empty spaces, and let $Y\leq X$ mean that Y is homeomorphic to a retract of X. The relation $\leq$ yields a partial ordering in Σ, and examples are given to show that there need not be an upper and/or lower bound; in particular, in Hilbert space the set of all compact n-dimensional AR ($n>0$) has no lower bound. It is noted that Σ always has an upper and lower bound in the (possibly) larger system Σ' obtained by adjoining to Σ the one-point space and the cartesian product of all members of Σ. Defining $X=Y$ if $Y\leq X$ and $X\leq Y$ gives an equivalence relation in Σ, the classes being the author's R-types. It is proved: (1) The set of R-types in the class of all finite polytopes is countably infinite. (2) The set of R-types in the class of all 2-dimensional compact AR in E^3 has the cardinal of the continuum c (this result implies that the R-types in the class of all compacta in Hilbert

space has cardinal c also). Calling X (right) adjacent to Y if $X \neq Y$ and X is an immediate successor of Y in the partial ordering of Σ, the index of proximity $I(X, Y)$, if finite, is defined as the number of intermediary spaces in a shortest chain of pairwise adjacent spaces connecting X and Y; $I(X, Y)=1$, if $X=S^2$, $Y=$projective plane, $\Sigma=$compacta, since the 2-disc is intermediary. A list of problems is given. *J. Dugundji* (Los Angeles, Calif.)

2974:

Sieklucki, K. On a family of power c consisting of R-uncomparable dendrites. Fund. Math. **46** (1959), 330–335.

The author complements a result of Borsuk [preceding review] by showing that the set of R-types in the class of all compact one-dimensional AR in the plane has cardinal of the continuum. *J. Dugundji* (Los Angeles, Calif.)

2975:

Deleanu, Aristide. Un théorème de point fixe pour les rétractes des espaces convexoïdes. C. R. Acad. Sci. Paris **247** (1958), 1950–1952.

The reviewer has demonstrated some fixed-point theorems on certain self maps, f, of punctured retracts of Tychonoff parallelotopes [Rev. Math. Pures Appl. **2** (1957), 371–374; MR **20** #1297]. The proofs involve a fixed point index Λ. The author (a) replaces the parallelotope by an acyclic convexoide space E, but needs to add the hypothesis that $f|D_t$ admits an extension to $\bar{D}_t$, where D_t is a punched-out open set of E. The demonstrations are those of the reviewer, where now Leray's index is understood for Λ. The author (b) indicates that a convexoide space need not be an ANR by an argument based on the unit sphere in a "convenable" L_2 space taken in the weak topology. {The author overlooked the fact that precisely this proof appears in another paper of the reviewer's [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 435–440; MR **17**, 1120].} *D. G. Bourgin* (Urbana, Ill.)

ALGEBRAIC TOPOLOGY

See also 2598, 2680a-b, 2972, 3027, 3028.

2976:

Whittlesey, E. F. Classification of finite 2-complexes. Proc. Amer. Math. Soc. **9** (1958), 841–845.

Let K be a finite connected simplicial 2-complex, every 1-simplex of which is incident with a 2-simplex. A point of K is "regular" or "singular" according as its open star is or is not a topological 2-cell. The singular points are classified according to the structures of their stars (books, cones, and so on). These points constitute a linear graph S whose complement is a set $\{M\}$ of bounded 2-manifolds. Then K is characterized by S (with specifications of singularities, incidences and orientations), the manifolds $\{M\}$, and certain paths and correspondences. The above restrictions on K are removed, to obtain a complete system of topological invariants for finite 2-complexes, partly numerical, partly of order. *S. S. Cairns* (Urbana, Ill.)

2977:

Eckmann, Beno; et Hilton, Peter J. Décomposition homologique d'un polyèdre simplement connexe. C. R. Acad. Sci. Paris **248** (1959), 2054–2056.

Continuing the theory described in earlier notes [same C. R. **246** (1958), 2444–2447, 2555–2558, 2991–2993; **247** (1958), 620–623; MR **20** #6694, #6695, #6696, #6697], the authors "dualise" the classification of fibre-spaces with fibre a $K(\pi, n)$; in fact: A "cofibration" is an inclusion $X \subset X'$ with the homotopy extension property; the quotient space $X'/X=X''$ is the "cofibre". A homotopy equivalence $X_1 \to X_2$ which is the identity on $X \subset X_1 \cap X_2$ is called an equivalence of the cofibrations (X, X_1), (X, X_2). If the cofibre is a Moore-space $K'(G, m)$ $(m \geq 3)$, the equivalence classes are in 1-1 correspondence with the "conjugacy" classes of elements of $\pi_{m-1}(G; X)=\Pi(R, X)$, where R is a given $K'(G, m-1)$, and $a, b \in \pi_{m-1}(G; X)$ are conjugate if there is an equivalence $q: R \to R$ such that $a=b \circ q$.

This theorem is then applied to develop a theory dual to that of Postnikov-systems. A simply connected polyhedron Y is called normal if it has an increasing sequence of subpolyhedra Y_m, $m=2, 3, \cdots$, such that the inclusion induces isomorphisms $H_n(Y_m) \to H_n(Y)=H_n$ $(n \leq m)$, while $H_n(Y_m)=0$, $n>m$. The cofibration $Y_m \subset Y_{m+1}$ is determined (up to equivalence class) by a conjugacy class of homologically trivial elements of $\pi_n(H_{n+1}; X_n)$; thus a relationship is set up between the normal polyhedron Y and a system $\mathscr{K}=(H_2, H_3, \cdots; k_2, k_3, \cdots)$, $k_n \in \pi_n(H_{n+1}; X_n)$. Conversely, such a system determines a space X, denoted by $K'(\mathscr{K})$; and, under a suitable equivalence relation, Y is equivalent to X.

Theorem: Every simply connected polyhedron is equivalent to a normal one and hence admits a system $\mathscr{K}$; two polyhedra with conjugate systems are equivalent. *V. Gugenheim* (Baltimore, Md.)

2978:

Chang, Su-Cheng. On proper isomorphisms of (μ, Δ, γ)-systems. I. Acta Math. Sinica **6** (1956), 270–301. (Chinese. English summary)

[Editor's note: Cf. the author's paper of the same title, Bull. Acad. Polon. Sci. Cl. III **4** (1956), 113–118; MR **19**, 972.]

A (μ, Δ, γ)-system

$$H[\{H^n\}, \{H^n(2)\}, \mu, \Delta, \gamma]$$

is a set of groups H^n, $H^n(2)$, $n=1, 2, \cdots$, and homomorphisms $\mu: H^n \to H^n(2)$, $\Delta: H^n(2) \to {}_2H^{n+1}$, $\gamma_p{}^n: H^n(2) \to H^{n+p}(2)$, $n=1, 2, \cdots$, where p is even and ${}_2H^{n+1}$ is a subgroup of H^{n+1} such that $2g=0$ if $g \in {}_2H^{n+1}$ [see Chang and Whitehead, Quart. J. Math. Oxford Ser. (2) **2** (1951), 167–174; MR **13**, 374]. A proper homomorphism g of a (μ, Δ, γ)-system H into another $\bar{H}$ is a set of homomorphisms either of H^n into $\bar{H}^n$ or of $H^n(2)$ into $\bar{H}^n(2)$ satisfying $g\mu=\bar{\mu}g$, $g\Delta=\bar{\Delta}g$, $g\gamma_p{}^n=\bar{\gamma}_p{}^n g$. Let $\mathscr{L}$ be the set of (μ, Δ, γ)-systems, each of which consists of no more than two non-trivial γ, say $\gamma_p{}^n$ and $\gamma_k{}^m$, with an additional condition that between the sets of groups $\{H^n, H^{n+1}, H^{n+p}, H^{n+p+1}\}$ and $\{H^m, H^{m+1}, H^{m+k}, H^{m+k+1}\}$ there is a unique common group. The author proves the following theorem: Among (μ, Δ, γ)-systems of $\mathscr{L}$, each properly isomorphic class is completely determined by Betti-numbers, torsions, block invariants and relative block invariants. *C. T. Yang* (Philadelphia, Pa.)

2979:

Chang, Su-Cheng. On proper isomorphisms of (μ, Δ, γ)-systems. II. Acta Math. Sinica 7 (1957), 295–308. (Chinese. English summary)

Following another paper of the author [see above], an $A_n{}^3$-system is a (μ, Δ, γ)-system consisting of groups H^r, $3 \leqq n \leqq r \leqq n+4$; $H^r(2)$, $n \leqq r \leqq n+3$; and homomorphisms μ, Δ, $n \leqq r \leqq n+3$, $\gamma_2{}^n$, $\gamma_2{}^{n+1}$ only, where $\Delta^{-1}(0) = \mu H^r$ and Δ has a right inverse. The main theorem is: The complete set of invariants of a proper isomorphism class of $A_n{}^3$-systems consists of Betti-numbers, torsions, block invariants, relative block invariants, characteristic polynomials and characteristic coefficients.

C. T. Yang (Philadelphia, Pa.)

2980:

Hu, Sze-Tsen. Algebraic local invariants of topological spaces. Compositio Math. **13**, 173–218 (1958).

The author introduces a method for defining algebraic local invariants at a point x_0 of a topological space X as follows. The tangent space $T(X, x_0)$ of X at the point x_0 is the subspace of the space of paths in X originating at x_0 consisting of those paths $\sigma(t)$ such that $\sigma(t) = x_0$ implies $t = 0$. A continuous map $f: X \to Y$ such that $f(x_0) = y_0$ and $f^{-1}(y_0) = x_0$ induces a map of $T(X, x_0)$ into $T(Y, y_0)$, and is called an admissible map. The homology, cohomology and homotopy groups of $T(X, x_0)$ (denoted by $L_q(X, x_0)$, $L^q(X, x_0)$ and $\lambda_q(X, x_0)$, respectively) are invariants of local character of X at x_0, and admissible maps induce appropriate homomorphisms among these algebraic invariants. It is shown that if X is locally triangulable at x_0, or X is the join of a space X_1 with the point x_0, then the homology groups of $T(X, x_0)$ give the same information as the local homology groups studied first by van Kampen [Thesis, Den Haag, 1929], and the generalizations of these studied by several authors. The advantage in the definition above lies in the fact that any theorem that can be applied to the homology, cohomology or homotopy groups of $T(X, x_0)$ gives immediately information of local character about the space X at the point x_0. Results in this paper obtained by this method include the following.

Properties of the functor $L_q(X, x_0)$ and maps induced by admissible maps corresponding to all of the Eilenberg-Steenrod axioms are established, with the exception of the excision axiom; also, there are established properties of $\lambda_q(X, x_0)$ corresponding to the axioms for homotopy, with the exception of the fibering axiom. A local map is an admissible map defined on some neighborhood of x_0. The degree of a local map $f: (X, x_0) \to (R^n, y_0)$ of a space X into n-dimensional Euclidean space is defined as the image of a generator of $L^{n-1}(R^n, y_0)$ under the induced homomorphism of the local cohomology groups

$$f^1: L^{n-1}(R^n, y_0) \to L^{n-1}(X, x_0).$$

There is obtained the local Hopf mapping theorem, i.e.: If X is locally triangulable at x_0, $n > 1$, dim X at $x_0 \leqq n$, then $f \to \deg f$ establishes a one-to-one correspondence between the homotopy classes of local maps of (X, x_0) into (R^n, y_0) and the elements of the local cohomology group $L^{n-1}(X, x_0)$. The author proves the local Hurewicz theorem [see Griffiths, Trans. Amer. Math. Soc. **89** (1958), 201–244; MR **21** #872], i.e.: If X is pathwise connected around x_0, then there is a natural homomorphism $h_n: \lambda_n(X, x_0) \to L_n(X, x_0)$ for each $n \geqq 1$. The homomorphism h_1 is an epimorphism, and its kernel is the commutator subgroup of the local fundamental group $\lambda_1(X, x_0)$. If $\lambda_q(X, x_0) = 0$ for every $q < n$, then h_n is an isomorphism. There is also stated a local version of the theorem of Whitehead which extends the Hurewicz theorem. There is established a local version of the theory of homotopy classes of maps, i.e.: The elements of $\lambda_n(X, x_0; \sigma)$ may be identified with the homotopy classes of local maps of $(R^{n+1}, 0)$ into (X, x_0) relative to σ (here σ refers to the point in $T(X, x_0)$ used as base point in computing homotopy groups).

In addition, certain theorems stated originally about local properties defined in other ways are proved using the invariants introduced here. The Poincaré-Bohl theorem: If $f: (U, x_0) \to (R^n, y_0)$ and $g: (V, x_0) \to (R^n, y_0)$ are two local maps of (X, x_0) into (R^n, y_0); and if there exists an open neighborhood W of x_0 in X such that $W \subset U \cap V$ and, for each $x \in W - x_0$, the line segment which joins $f(x)$ to $g(x)$ in R^n does not contain the point y_0; then we have $\deg(f) = \deg(g)$. There are theorems about stability of maps [see Hopf and Pannewitz, Math. Ann. **108** (1933), 433–465]. A map $f: X \to Y$ is unstable if for every open neighborhood V of x_0, there exists a homotopy $f_t: X \to Y$, $0 \leqq t \leqq 1$, such that $f_0 = f$, $f_1(X) \subset Y - y_0$, and $f_t(x) = f(x)$ for every $x \in X - V$, and every $t \in I$. It is proved that if X is a normal Hausdorff space, f, g two maps of (X, x_0) into (Y, y_0) which are locally homotopic, then if g is unstable, f is unstable. Also, if X is locally triangulable at x_0, Y locally Euclidean at y_0, dim X at $x_0 \leqq$ dim Y at y_0, dim Y at $y_0 > 1$, then a local map f of (X, x_0) into (Y, y_0) is unstable if and only if $\deg(f) = 0$.

The relation of the condition that $T(X, x_0)$ be pathwise connected to other conditions is examined. It is proved, among other things, that if a space X has countable basis at a point $x_0 \in X$ and if x_0 is a local homotopy non-r-cut point of X for $r = 0$, 1, then $T(X, x_0)$ is pathwise connected. The implications of the condition that $\lambda_n(X, x_0; \sigma) = 0$ are examined. For example, if X is the join of the point x_0 and a space X_1, then x_0 is a local homotopy non-n-cut point if and only if $\lambda_n(X, x_0; \sigma) = 0$ for every basic path $\sigma \in T(X, x_0)$. Finally an application to the theory of semi-groups is made, namely, it is shown that two theorems of Wallace [see pp. 96–97, Bull. Amer. Math. Soc. **61** (1955), 95–112; MR **16**, 796] can be proved by the methods of this paper.

T. R. Brahana (Athens, Ga.)

2981:

Sugawara, Masahiro. Some remarks on homotopy equivalences and H-spaces. Math. J. Okayama Univ. **8** (1958), 125–131.

Let $f: X \to Y$ and $g: Y \to X$ be mutually inverse homotopy equivalences, so that deformations $F_t: X \to X$ and $G_t: Y \to Y$ exist such that $F_0 = 1$, $F_1 = gf$ and $G_0 = 1$, $G_1 = fg$. The author observes that fF_t and $G_t f$ are both deformations of f into fgf, and that gG_t and $F_t g$ are both deformations of g into gfg. If X and Y are CW-complexes it is proved that g, F_t and G_t can be chosen so that fF_t and $G_t f$, also gG_t and $F_t g$, are equivalent deformations. Then this is used to improve a result in a paper by the author [same J. **7** (1957), 123–149; MR **20** #3546] and hence obtain a necessary and sufficient condition for H-spaces to be homotopy-associative.

I. M. James (Oxford)

2982:

Halpern, Edward. Twisted polynomial hyperalgebras. Mem. Amer. Math. Soc. no. 29, 61 pp. (1958).

This memoir gives an algebraic treatment of certain ideas which arose in the study of H-spaces.

If A is a commutative ring with unit, an "A-hyperalgebra" is a unitary A-module, H, with product and coproduct, $\phi: H \otimes H \to H$ and $\Delta: H \to H \otimes H$, respectively, subject to the condition $\Delta\varphi = (\varphi \otimes \varphi)\rho(\Delta \otimes \Delta)$ where ρ permutes the second and third factors. The introductory chapter gives the usual definitions: associativity, augmentation, gradation, subhyperalgebra, dual hyperalgebra. Primitive elements are defined, and in the case where H is torsion-free and of finite type, they are characterized as orthogonal to the decomposable elements of the dual hyperalgebra. Assume below that H is associative with unit, and when graded, it is augmented by a standard augmentation.

Chapters 2 and 3 study monogenic twisted polynomial algebras and hyperalgebras (the above assumptions also being made for A-algebras). Let $x_0, x_1, \cdots$ be a sequence of h elements, $2 \leq h \leq \infty$. Let $t = t_{m,n}$ be a bisequence, defined when $0 \leq m+n < h$, which is symmetric and satisfies $t_{0,n} = t_{m,0} = 1$ and $t_{m,n}t_{m+n,k} = t_{m,n+k}t_{n,k}$. The ring A will be assumed to be an integral domain. Then, the free A-module generated by the $\{x_i\}$, with multiplication defined by $x_m x_n = t_{m,n}x_{m+n}$ if $m+n<h$, $=0$ otherwise, is called a "monogenic twisted polynomial A-algebra" of height h and type $t = \{t_{m,n}\}$. One easily sees that $x_1^m = t_{1,0}\cdots t_{1,m-1}x_m$, thus motivating the name. If $t_{m,n} = (m, n)\cdot 1$, where (m, n) is the binomial coefficient, then the algebra is said to be of binomial type. If $t_{m,n} | (m, n)\cdot 1$, it is said to be of pseudobinomial type. Calling two twisted polynomial algebras type-equivalent if they have the same height, and the corresponding elements of the bisequences differ by unit factors, the author classifies monogenic, twisted polynomial hyperalgebras, under certain reasonable restrictions.

The author next defines pseudogenerators; a subset X of an A-algebra H is said to "pseudogenerate" H if the integral closure of the subalgebra generated by X is H. This concept is important for the final results of the paper. In the case of torsion-free, graded, anticommutative hyperalgebras, the existence of primitive pseudogenerators implies that the coproduct is anti-commutative.

Enter twisted polynomial hyperalgebras. Let H be a torsion-free, graded, anticommutative A-algebra (A an integral domain). Let $X \subset H$ be a set of homogeneous elements of positive degree. Each $x \in X$ pseudogenerates a graded subalgebra, written $J[x, h(x), t(x)]$. If the map $i: X \to H$ induces an algebra isomorphism of the weak tensor product of all $J[x, h(x), t(x)]$ with H, then H is called a "twisted polynomial algebra" in the elements of X. When A has characteristic p or 0, and H is a graded hyperalgebra, the author gives the height of any element and also an explicit formula for the coproduct. He also gives necessary conditions for two such $J[X, h, t]$ and $J[Y, h', t']$ to be the same algebra. ($J[X, h, t]$ denotes the twisted polynomial algebra in the elements of X.)

After a brief section on duality, one reaches the heart of the matter: the study of certain minimality properties of systems of pseudogenerators. The point of departure is A. Borel, Ann. of Math. (2) **57** (1953), 115–207 [MR **14**, 490]. The author adopts a notion essentially equivalent to minimality. This is the idea of a "special system of pseudogenerators", defined by requiring that $i: X \to H$ induce an algebra isomorphism of the weak tensor product of the $A[x]/(x^{h(x)})$ onto $P_A(X)$. Again in the case of torsion-free, graded, anti-commutative hyperalgebras, he gives sufficient conditions for H to be a $J[X, h, t]$.

The style of the memoir is good, and the text is well illustrated with examples.

D. W. Kahn (Cambridge, Mass.)

2983:

O'Neill, Barrett. Induced homology homomorphisms for set-valued maps. Pacific J. Math. **7** (1957), 1179–1184.

For any continuous set-valued function $F: X \to Y$ of compact metric spaces, the author defines a vector space $\{H\}$ of "induced homomorphisms" of $H(X)$ into $H(Y)$, $H(X)$ denoting Čech homology with coefficients in a field. He establishes certain naturality and non-triviality properties and proves the theorem: Let F be a set-valued self-map on a compact polyhedron X and n an integer such that for each $x \in X$, $F(x)$ consists either of one or of n homologically trivial components. Then there is an $h \in \{H\}$ such that if $\Lambda(h) \neq 0$, F has a fixed point. Further, if X is homologically trivial, then F has a fixed point. ($\Lambda(h)$ is the Lefschetz number $\sum (-1)^q$ (trace h^q).) The case $n = 1$ is the polyhedral case of a theorem of Eilenberg and Montgomery [Amer. J. Math. **68** (1946), 214–222; MR **8**, 51].

E. Dyer (Chicago, Ill.)

2984:

Smale, Stephen. A classification of immersions of the two-sphere. Trans. Amer. Math. Soc. **90** (1958), 281–290.

An immersion of one C' differentiable manifold in another is a regular map, and a regular homotopy of an immersion is a homotopy through immersions which induces a homotopy of the tangent bundle. The author solves the problem of classifying C^2-immersions of S^2 in E^n, $n \geq 3$, by regular homotopy.

Let f, g be two C^2-immersions of S^2 in E^n. By deforming g if necessary we may suppose that f, g agree on some closed neighbourhood U of a point $x_0 \in S^2$. Then $D = S^2 - \text{int } U$ is a topological disk on which we can suppose given a fixed field of 2-frames, and f, g induce thereby maps of D into the Stiefel manifold $V_{n,2}$ which agree on the boundary $\dot{D}$. Thus a difference element $\Omega(f, g) \in \pi_2(V_{n,2})$ is defined (we need not worry about base points since $V_{n,2}$ is 1-connected—unless $n = 3$, when $\pi_2 = 0$). The author proves (theorem A) that f, g are regularly homotopic if and only if $\Omega(f, g) = 0$, and that, given f and $\Omega \in \pi_2(V_{n,2})$, there exists g with $\Omega(f, g) = \Omega$. A sharper result is obtained if $n = 4$, from which it follows that two C^2-immersions of S^2 in E^4 are regularly homotopic if and only if they have the same algebraic intersection number. The author also gives the generalization of the main theorem to the case where E^n is replaced by an arbitrary C^2 manifold M^n.

The methods used are extensions of those of a previous paper [Trans. Amer. Math. Soc. **87** (1958), 492–512; MR **20** #2[illegible]]; in particular they enable the author to show that any regular map of $\dot{D}$ into E^n may be extended to D.

P. J. Hilton (Birmingham)

2985:

Nettleton, R. E.; Goldberg, K.; and Green, M. S. Dense subgraphs and connectivity. Canad. J. Math. **11** (1959), 262–268.

An "isthmus" of a connected graph G is a minimal complete subgraph whose removal disconnects G. An "articulator" of G is a subgraph H which is not complete, whose removal disconnects G, and such that each connected component of $G-H$ contains a neighbor of each point of H. In theorem 3.1, the authors note that a subgraph H is either an articulator or an isthmus if and only if H is a minimal subgraph whose removal disconnects G. Hence one could take this theorem as the definition of a "separating subgraph" and then describe an isthmus or articulator as a complete or incomplete separating subgraph respectively.

A connected subgraph H of G is "dense" if $H=G$ or if every point in $G-H$ has a neighbor in H. The authors note in theorem 2.2 that the collection of all the dense subgraphs of a connected graph constitutes a lattice under inclusion. There are several other observations on dense minimal and dense maximal subgraphs. For example, theorem 3.8 shows that the intersection of all dense subgraphs is exactly the subgraph containing all the cutpoints of G. The main theorem, theorem 3.4, gives 7 consequences of the hypothesis that a graph G be n-connected. Some of these have already been given in the well-known paper of H. Whitney [Amer. J. Math. **54** (1932), 150–168]. There is no mention of Whitney's paper, nor are either of the two references (one to a paper of the reviewer and the other to the standard book on graph theory by König) mentioned at all in the text of the article.

F. Harary (Ann Arbor, Mich.)

2986:

Harary, Frank. An elementary theorem on graphs. Amer. Math. Monthly **66** (1959), 405–407.

A block of a connected graph G is defined as a maximal connected subgraph containing no cut points of itself. Let the number of blocks of G be N, and let n_i be the number of components of the graph obtained by removing the ith cut point. The author proves that

$$N-\sum_{i=1}^{r} n_i+r=1.$$

W. T. Tutte (Toronto, Ont.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 2749, 3114, 3199.

2987:

Fabricius-Bjerre, Fr. On linearly monotone elementary curves. Nordisk Mat. Tidskr. **7** (1959), 27–35. (Norwegian)

An elementary curve in E^2 consists of a finite number of differentiable convex arcs joining with the same tangent, cusps being allowed. The only other singularities are therefore inflections and multiple points. A curve $p(t)$, $a\leq t\leq b$, is linearly monotone if each line intersects it in a finite number of points $p(t_1)$, $\cdots$, $p(t_n)$, $n\geq 1$, such that either $t_1<t_2<\cdots<t_n$ or $t_1>t_2>\cdots>t_n$. A linearly monotone elementary curve has no multiple points and only cusps of the first kind. If the curve has no cusps and a line intersects it in n points, then it has at least $n-2$ inflections. The principal result characterises the linearly monotone curves among the elementary curves by the property that through each ordinary point a line other than the tangent exists which intersects the curve only at that point.

H. Busemann (Los Angeles, Calif.)

2988:

Lee, Shen-ling. Vertices of generalized regular plane curves. Acta Math. Sinica **8** (1958), 305–323. (Chinese. English summary)

A plane smooth curve which is twice continuously differentiable except at n non-regular points P_0 is defined as a generalized plane regular curve if: (1) a neighborhood of each P_0 lies on one side of the tangent at P_0, (2) on a deleted neighborhood of P_0 the curvature is monotonic, but not constant, such that its left-hand limit $\geq$ or $\leq$ its right-hand limit at P_0. A non-regular point P_0 is called a vertex if the curvature in its deleted neighborhood is less [greater] than the smaller [larger] one of the two limits at P_0. An arc is defined as a vertex if along the arc the curvature is a constant less [greater] than the curvatures at the regular points immediately preceding and following the arc.

The author proves that the four-vertex theorem and other related results about a plane regular oval and a plane regular simple closed curve hold for his generalized plane regular oval and generalized plane regular simple closed curve.

T. K. Pan (Norman, Okla.)

2989:

Gheorghiev, Gh. Sulla geometria dei coni dell' S_3 euclideo. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **3** (1957), 125–132. (Russian and Romanian summaries)

A set of unitary vectors defined in a domain M of an S_3 by means of a vector function $I(M,t)$ depending on a scalar argument gives rise to a cone of directions in each point of M. The generators of these cones form a manifold the geometry of which has been studied by geometers from various points of view. In this paper the author applies the method of moving frames of Cartan. As a result the fundamental invariants of the manifold are described in terms of Pfaffian forms the coefficients of which have a definite geometrical meaning. In this way some new properties of the manifold are found, and the generalization of results referring to special cases is made. Some special cases, such as that of cones of revolution, the case of a Lie manifold, are discussed in detail.

G. Soós (Debrecen)

2990a:

Nitsche, Johannes C. C. On harmonic mappings. Proc. Amer. Math. Soc. **9** (1958), 268–271.

2990b:

Nitsche, Johannes C. C. On an estimate for the curvature of minimal surfaces $z=z(x,y)$. J. Math. Mech. **7** (1958), 767–769.

2990c:

Nitsche, Johannes C. C. A remark on Enneper's minimal surface. Amer. Math. Monthly **66** (1959), 295–297.

The first two of these papers contain applications of the idea of the author's earlier papers [Arch. Math. **7** (1957), 417–419; Ann. of Math. (2) **66** (1957), 543–544; MR **19**, 448, 878]. In the former, the author obtains a new demonstration of a lemma of Bers, that there exists no schlicht harmonic mapping of the unit disc onto the entire plane, and gives also an improvement of an estimate of E. Heinz, that in a 1-1 harmonic mapping of the unit disc $|\gamma|<1$ onto the unit disc $|z|<1$ with $z(0)=0$, there exists a positive μ independent of the particular mapping considered, such that $|\nabla z|^2_{\gamma=0}\geqq\mu$. The estimate $\mu\geqq 0.358$ of Heinz is improved by the author to $\mu\geqq 0.5$. The author states that another proof that $\mu\geqq 0.5$ has been communicated to him by H. Hopf. In a note, "added in proof", he states that a refinement of his own method yields $\mu\geqq 0.64$.

In the second paper the author considers the estimate $|K|_{x=y=0}\leqq C/R^2W_0^2$ known to be valid for the Gaussian curvature K of a minimal surface $z=z(x, y)$ defined in $x^2+y^2<R^2$. Here $W_0^2=[1+z_x^2+z_y^2]_{x=y=0}$. This inequality was first given by E. Heinz [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 51–56; MR **14**, 885], who found $C\leqq 3\pi^2W_0^2$. It was shown by E. Hopf [J. Rational Mech. Anal. **2** (1953), 519–522; MR **14**, 1119] that $C\leqq 8/\mu$. In the present paper, the author obtains $C\leqq 49/4$. He compares this with the estimate of E. Hopf, using the value $\mu\geqq\frac{1}{2}$ obtained in the paper referred to above. In this case Hopf's estimate becomes $C\leqq 16$. {However, if the value claimed by the author, $\mu\geqq 0.64$, is used in Hopf's estimate, we obtain $C\leqq 50/4$. The reviewer remarks that in a recently completed work R. Osserman has proved the result $C\leqq 8$ [see #2991].}.

In the third paper the author shows by example that $C\geqq 1.05$. The example considered is the (explicitly known) minimal surface of Enneper, and the estimate follows directly from a lemma that this surface covers simply the circle $x^2+y^2<64/243$. It has been remarked by R. Osserman that the radius R_s of the largest circle which is covered simply can be obtained exactly by considering the set of points (x, y) for which the normal to the surface is horizontal. This observation yields $R_s=\frac{8}{3}$ and implies the better estimate $C\geqq 16/9$. *R. Finn* (Stanford, Calif.)

2991:

Osserman, Robert. An analogue of the Heinz-Hopf inequality. J. Math. Mech. **8** (1959), 383–385.

Suppose S is a piece of a minimal surface $z=f(x, y)$ having a single-valued projection onto the circle $x^2+y^2<\rho^2$. Then, as was first proved by E. Heinz [op. cit., #2990b above], the Gaussian curvature K_0 at the origin cannot in absolute value be greater than $C_1\cdot\rho^{-2}$. E. Hopf [op. cit., ibid.] improved this estimate to $|K_0|\leqq C_2W_0^{-2}\rho^{-2}$, where $W_0=[1+f_x^2(0, 0)+f_y^2(0, 0)]^{1/2}$. The best numerical value for the constant C_2 known so far was given by the reviewer [#2990b] as 49/4. The author generalizes these results to a simply connected piece S of a minimal surface whose spherical image omits a whole neighborhood of a point. (In the case $z=f(x, y)$ the spherical image covers only a semisphere.) Using his previous results [Comm. Pure Appl. Math. **12** (1959), 229–232; MR **21** #4436] he proves the theorem: Let the normals to S form an angle greater than $\alpha>0$ with a fixed direction. Let p be any point of S, $K=K(p)$ the Gaussian curvature at p, d the geodesic distance from p to the boundary of S, and ω the angle between the normal at p and the fixed direction. Then

$$|K|\leqq d^{-2}[2(R^2-r^2)(1+R^2)R^{-1}(1+r^2)^{-2}]^2.$$

Here $R=\cot\alpha/2$ and $r=\cot\omega/2$. {Reviewer's remark: In the meantime [Stanford Univ. Tech. Rep. no. 11, 1959] the author has improved his estimate and has, in particular, found the exact estimate for the case $r=0$; namely $|K|\leqq d^{-2}[\frac{8}{3}R(3+R^2)]^2$. As a consequence, for a minimal surface $z=f(x, y)$ he can improve the numerical value of the constant C_2 in the case $r=0$ (that is, $W_0=1$) to 64/9 and for general W_0 to 8.}

J. C. C. Nitsche (Minneapolis, Minn.)

2992:

Kamalamma, K. N. The congruence of tangents to a system of curves on a surface. Compositio Math. **13**, 247–256 (1958).

The author investigates the congruence T of tangents to a given system C of curves on a given surface. Properties of T are obtained when C is a system of: general curves, geodesic parallels, lines of curvature, asymptotic lines. The methods used are those of conventional differential geometry. *T. R. Hollcroft* (Aurora, N.Y.)

2993:

Bereis, R.; und Brauner, H. Über koaxiale euklidische Schraubungen. Monatsh. Math. **61** (1957), 225–245.

Verf. betrachten die involutorischen Korrelationen, welche die Gesamtheit der ∞^3 Schraublinien mit gemeinsamer Achse a mit der Gesamtheit der Schraubtorsen um a vertauschen. Die einzigen linearen Nullsysteme dieser Eigenschaft sind die Nullsysteme der Normalengewinde der Schraubungen. Die einzigen Flächen 2. Grades, deren Polarsysteme die Eigenschaft haben, sind die Drehparaboloide um a und die gleichseitigen hyperbolischen Paraboloide mit a als Achse. Es werden noch die Schnittkongruenzen des Schraubtangentenkomplexes mit dem Gewinde des Nullsystems und mit dem Tangentenkomplex der Paraboloide untersucht. *O. Bottema* (Delft)

2994:

Strubecker, Karl. Über Komplexflächen bei euklidischen Schraubungen. Monatsh. Math. **62** (1958), 297–323.

Bereis und Brauner [#2993] haben die Kongruenz jener Bahntangenten einer euklidischen Schraubung untersucht, welche ein zur Schraubung koaxiales gleichseitiges hyperbolisches Paraboloid berühren. Verf. weist darauf hin, dass ihre Resultate Sonderfälle viel allgemeinerer Sätze sind, welche er schon früher [A. Frey und K. Strubecker, J. Reine Angew. Math. **193** (1954), 209–238; **194** (1955), 1–20; MR **16**, 745; **17**, 188] gewonnen hat und die es auch ermöglichen, weitere Fragen zu beantworten. So zeigt er, dass die zweite Brennfläche der Kongruenz 9.ter Ordnung ist und er gibt ihre Gleichung an. Verf. setzt dann seine Methode ausführlich auseinander unter Benutzung eines eleganten analytischen Apparates mittels komplexer Zahlen. Der Bahntangentenkomplex der Schraubung gestattet eine dreigliedrige Gruppe von Kollineationen, welche für die Punkte P des Raumes transitiv ist. Wandert P längs einer Kurve oder

Fläche, dann wird dadurch eine ein- bzw. zweigliedrige Schar von "Komplexschiebungen" erklärt, bei der alle anderen Raumpunkte "kongruente" Kurven bzw. Flächen beschreiben. Diese Idee wird dann auf (besondere) Komplexkurven und Flächen angewandt und führt zu sehr allgemeinen Beziehungen. Nachdem auch nichtprojektive Automorphien des Komplexes in Betracht gezogen sind erreicht Verf. sehr allgemeine Sätze für die Kongruenz der Bahntangenten der Punkte einer ausgedehnten Klasse von Komplexflächen. *O. Bottema* (Delft)

2995:

Terracini, Alessandro. Su alcuni sistemi di elementi curvilinei. Boll. Un. Mat. Ital. (3) **13** (1958), 395–405. (English summary)

In an n-dimensional projective space S_n, through a point, a curvilinear element E_2 of the second order defined by differential equations of the type

$$\frac{d^2x_i}{dx^2} = A_i + \sum_{j=1}^{n-1} B_{ij}\frac{dx_j}{dx} + \sum_{j,m=1}^{n-1} C_{ijm}\frac{dx_j}{dx}\frac{dx_m}{dx} + \frac{dx_i}{dx}\sum_{j,m=1}^{n-1} P_{jm}\frac{dx_j}{dx}\frac{dx_m}{dx} \quad (i = 1, \cdots, n-1)$$

is called a quasigeodesic form; where $x_1, \cdots, x_{n-1}$ are unknown functions of x, and A_i, B_{ij}, C_{ijm}, P_{jm} are given functions of $x, x_1, \cdots, x_{n-1}$. In an S_n an E_2 through the origin O expressed in terms of nonhomogeneous projective coordinates $x_0 = x, x_1, \cdots, x_{n-1}$ by

$$x_i = a_i x + b_i x^2 + [3] \quad (i = 1, \cdots, n-1)$$

is quasigeodesic if and only if it can be carried into itself by a point transformation. In S_3 a composed curvilinear element $E_{2,1}$ is defined to be expressed in terms of nonhomogeneous projective coordinates x, y, z by

$$y = a_{11}x^2 + a_{12}x^3 + [4], \quad z = a_{22}x^3 + a_{23}x^4 + [5],$$

where the a's are given constants and $a_{11}a_{22} \neq 0$. If

$$a_{12} = k_0 a_{22} + k_1 a_{11} + 2k_2 a_{11}^2, \quad a_{23} = 2k_1 a_{22} + 3k_3 a_{11} a_{22},$$

the k's being given constants, then the $E_{2,1}$ is called a Δ system. In particular, if $k_2 = k_3$, then it is called a Γ system. Geometric characterizations of Γ systems are also given. *C. C. Hsiung* (Madison, Wis.)

2996:

Terracini, Alessandro. Su certi sistemi ∞^1 di linee spaziali. Boll. Un. Mat. Ital. (3) **13** (1958), 564–573. (English summary)

Continuing the work in the paper reviewed above, the author obtains a system of two differential equations representing a Γ (or Δ) system of space curves. Examples are given to show that general space cubics and circular helices form two Γ systems, and that certain special space cubics and cuspidal quartics form two Δ, but not Γ, systems. *C. C. Hsiung* (Madison, Wis.)

2997:

Mihăileanu, N. Dualité entre le calcul de coordonnées et de congruences. Com. Acad. R. P. Romîne **8** (1958), 141–145. (Romanian. Russian and French summaries)

In this paper a comparison is made between the differential calculus in terms of coordinate systems and that based on the use of congruences. The author observes a type of duality according to which every formula gives a valid statement in the terminology of the other system of calculus if every quantity occurring in it is replaced by its dual equivalent. *G. Soós* (Debrecen)

2998:

Penzov, Yu. E. Two-dimensional bundles of geometric objects in X_1. Mat. Sb. N.S. **46 (88)** (1958), 291–314. (Russian)

In a previous paper [same Sb. **26 (68)** (1950), 161–182; MR **12**, 52] types of two-component differential geometric objects of class v were studied from a local point of view. Here they are studied in the large. The starting point is the work of Haantjes and Laman [Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 208–222; MR **15**, 990], from which is taken the concept of a bundle of geometric objects of class v of constant type h in the differentiable manifold X_n of class r ($r \geq v$). Every closed subgroup $\mathfrak{H}$ of the full differential group $\mathfrak{D}^{(v,n)}$ determines a bundle of transitive geometric objects of class $\leq v$ of constant type in X_n. First $\mathfrak{D}^{(v,1)}$ is considered, together with its algebra; then all subalgebras of dimension $v-2$. Then the case of three-dimensional closed groups of $\mathfrak{D}^{(5,1)}$ (the $\mathfrak{D}^{(v,1)}$ for $v \geq 6$ have no subgroups of dimension $v-2$) is taken up, with the corresponding two-dimensional transitive bundles of geometric objects. A full classification of these bundles of objects of class $\leq v$ of constant type in X_1 is reached. *D. J. Struik* (Cambridge, Mass.)

2999:

Ruscior, Ştefania. Sur une correspondance par incidence entre deux variétés réglées, dans S_4. Bul. Inst. Politehn. Iaşi (N.S.) **4 (8)** (1958), 45–48. (Romanian. Russian and French summaries)

In an affine S_4 the author defines a special correspondence between two ruled surfaces V and V^* with respect to a surface S. A generator g of V is said to correspond to g^* of V^* if their intersection is a point M on S. Then a map of the points of g onto those of g^* is defined in such a way that the tangent hyperplanes in corresponding points intersect the tangent plane at M of S in the same line. The resulting map of V onto V^* is called a correspondence by incidence. The paper contains some results concerning the properties of such a correspondence. *G. Soós* (Debrecen)

3000:

Nožička, F. Zur Berührung von zweien Mannigfaltigkeiten in einem affin-euklidischen Raum. Ann. Polon. Math. **6** (1959/60), 83–86.

L'extrait (sans démonstrations) d'article déjà rapporté [Časopis Pěst. Mat. **83** (1958), 171–201; MR **20** #2740]. *A. Švec* (Prague)

3001:

Marcus, F. Sur un problème de E. Čech. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **9** (1958), no. 1, 73–100. (Romanian. Russian and French summaries)

Sur une surface (non de coïncidence) de S_3 il pourrait exister (tout au plus) quinze familles ∞^1 de courbes pangéodésiques planes (p. p.), mais il n'y a pas de surfaces

avec trois (deux) familles de courbes p. p. sans que les courbes de deux familles ni soient pas conjuguées. Les surfaces de coïncidence les plus générales ne possèdent aucune famille de courbes p. p. On étudie les courbes p. p. données par l'équation $du/dv = \text{const.}$, et l'existence des surfaces avec une famille de courbes p. p.

A. Švec (Prague)

3002:

Švec, Alois. Projektive Deformation der Strahlenkongruenzen in mehrdimensionalen Räumen. Rev. Math. Pures Appl. **1** (1956), no. 3, 91–92.

The concept of point deformation of congruences was introduced by E. Čech [Czechoslovak Math. J. **5** (**80**) (1955), 234–273; MR **17**, 658]. In a previous paper [ibid. 546–558; MR **18**, 231], the author has shown that in space S_n, $n \geqq 5$, a projective deformation of the second order is equivalent to a point deformation. In that paper he also treated the case $n = 4$. In the present paper he studies line congruences in S_5 by replacing the concept of deformation with that of bideformation. He finds that if two congruences are in both torsal correspondence and bideformation, they are also in point deformation, hyperplane deformation, and in two types each of focal deformation and asymptotic deformation.

T. R. Hollcroft (Aurora, N.Y.)

3003:

Mayer, O. Sur certains systèmes triplement conjugués. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **9** (1958), no. 1, 113–124. (Romanian. Russian and French summaries)

A 3-conjugate system of planes in a projective P_n, according to C. Guichard [Scientia. Ser. Phys.-Math. **25** (1905)], is formed by the tangent planes to a 3-conjugate point system of surfaces, which is a system of Laplacean surfaces $x_i = x_i(u_1, u_2, u_3)$, or surfaces satisfying three Laplacean equations

$$\partial_{jk}x = a_i x + b_i \partial_j x + c_i \partial_k x;$$

here i, j, k are circularly permutable on 1, 2, 3. The system of planes is defined by its so-called focal triangle z_1, z_2, z_3, all functions of the u_i; if $\partial_j z_i = \beta_{ij} z_j$, then the equations $t_i = \beta_{ki} z_j - \beta_{ji} z_k$ define the so-called secondary foci. The paper studies the cases (a) in which the foci t_i are collinear, and (b) in which the lines $z_1 t_1$, $z_2 t_2$, $z_3 t_3$ are concurrent; also those 3-conjugate systems x_i which admit 3-conjugate systems of incident planes $z_1 z_2 z_3$ such that the cross ratios $[x z_i x_i^k x_i^j]$ are equal to the same constants: here $x_i^k = \partial_i x - b_j x$, $x_i^j = \partial_i x - c_k x$.

D. J. Struik (Cambridge, Mass.)

3004:

Cohen, Paul J. On Green's theorem. Proc. Amer. Math. Soc. **10** (1959), 109–112.

The author proves

$$\int_C A dx + B dy = \iint_Q \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy$$

for a rectangle Q under the following assumptions. $A(x, y)$, $B(x, y)$ are defined and continuous in the closed rectangle. In the interior the four derivatives $\partial A/\partial x$, $\partial A/\partial y$, $\partial B/\partial x$, $\partial B/\partial y$ exist everywhere except in a countable number of points. The surface integral exists Lebesgue. The emphasis of the theorem lies in the fact that $A(x, y)$, $B(x, y)$ are not assumed to have total derivatives, but only the partial ones.

S. Bochner (Princeton, N.J.)

3005:

Ahn, Jae Koo. A correspondence on Laguerre spaces. Kyungpook Math. J. **1** (1958), 61–64.

Let L^n and $\bar{L}^n$ be Laguerre spaces [as studied by Y. Tomonaga, J. Math. Soc. Japan **2** (1951), 253–266; **3** (1951), 310–316; MR **13**, 158, 983; see also Sūgaku **2** (1950), 297–311; MR **13**, 491]. Let there be a one-parameter family of hyperspheres invariant in L^n and one in $\bar{L}^n$. Let the underlying Riemann spaces for L^n and $\bar{L}^n$ satisfy a certain condition (which forces them to have metric tensors of a specified type). Then the author presents a necessary and sufficient condition that the two one-parameter families of fixed hyperspheres shall correspond.

A. Schwartz (New York, N.Y.)

3006:

Havlíček, Karel. Survey of the fundamental concepts of the geometry of curved spaces. Pokroky Mat. Fys. Astr. **3** (1958), 639–659. (Czech)

An expository article concerning the basic concepts of the geometry of curved spaces. The author uses his pedagogical skill to familiarize the reader with the classical concepts of connections, metric tensor and metric connections. The last section is devoted to a derivation of Levi-Civita parallelism on a surface.

V. Hlavatý (Bloomington, Ind.)

3007:

Petrescu, Ştefan. Considérations intrinsèques dans les espaces X_n à connexion métrique. I. Bull. Math. Soc. Sci. Math. Phys. Roumaine (N.S.) **1** (**49**) (1957), 195–221.

W. C. Graustein [Trans. Amer. Math. Soc. **36** (1934), 542–585] studied Riemannian spaces with the use of the so-called ennuple of congruences. The author of the paper under review studies the spaces with metric connexion with the same method. A space with metric connexion is a space with Riemannian metric $ds^2 = g_{ji} d\xi^j d\xi^i$ in which a metric connexion Γ_{ji}^h, that is, a connexion such that $\nabla_k g_{ji} = \partial_k g_{ji} - \Gamma_{kj}^a g_{ai} - \Gamma_{ki}^a g_{ja} = 0$, is introduced.

K. Yano (Tokyo)

3008:

Matsumoto, Makoto. S-extensions of Riemannian spaces. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 107–118.

Let V_n be an n-dimensional manifold with Riemannian metric $ds^2 = g_{ji} d\xi^j d\xi^j$ and with a non-symmetric metric connexion L_{ji}^h; then we have

$$\nabla_k g_{ji} = \partial_k g_{ji} - L_{kj}^a g_{ai} - L_{ki}^a g_{ja} = 0,$$

from which

$$L_{ji}^h = \{_j^h{}_i\} + S_{ji}^h - S_{ji}^h - S_{ij}^h, \tag{1}$$

where $S_{ji}^h = \frac{1}{2}(L_{ji}^h - L_{ij}^h)$ is the torsion tensor and $S^h{}_{ji} = S_{bj}^a g^{bh} g_{ai}$.

Equation (1) shows that the symmetric part $\Gamma_{ji}^h = \frac{1}{2}(L_{ji}^h + L_{ij}^h)$ of L_{ji}^h coincides with the Christoffel symbols $\{_j^h{}_i\}$ if and only if (2) $S^h{}_{ji} + S^h{}_{ij} = 0$, that is, if and only if

the covariant torsion tensor $S_{ji\hbar}=S_{ji}{}^{a}g_{a\hbar}$ is skew-symmetric in all its indices. The condition (2) is called an S-condition and the space with a metric connexion satisfying the S-condition is said to be of S-connexion.

Let R_n be a Riemannian space with the metric $ds^2=g_{ji}d\xi^j d\xi^i$ and define the connexion in R_n by $L_{ji}{}^{h}=\{{}^{h}_{ji}\}+S_{ji}{}^{h}$, where $S_{ji\hbar}$ is a skew-symmetric tensor; then $L_{ji}{}^{h}$ is a metric S-connexion. A space with such S-connexion is called an S-extension of the original Riemannian space.

The author proves the following theorems.

The scalar curvature of any S-extension of a Riemannian space with positive definite metric is not greater than the scalar curvature of the original Riemannian space.

If the tensor $S_{ji\hbar}$ is harmonic, then the Ricci tensor of the S-extension is symmetric.

If the tensor $S_{ji\hbar}$ is a Killing tensor, then the curvature tensor $L_{kji\hbar}$ of the S-extension satisfies $L_{kji\hbar}=-L_{kji\hbar}=-L_{jki\hbar}$, $L_{kji\hbar}=L_{i\hbar kj}$, and the Ricci tensor L_{ji} is symmetric.

If a Riemannian space is completely harmonic, then any S-extension of the space is also completely harmonic.

In the last section the author studies subspaces in a metric space with an S-connexion. *K. Yano* (Tokyo)

3009:

Dolbeault-Lemoine, Simone. Réductibilité de variétés plongées dans un espace à courbure constante. Rend. Mat. e Appl. (5) **17** (1958), 453–474.

A Riemannian manifold V is said to be locally reducible if its restricted homogeneous holonomy group is reducible or, more geometrically, if V is locally a product of two Riemannian spaces. After a brief introduction (ch. I), the author studies locally reducible V_{n-1} imbedded in a space V_n of constant curvature (ch. II). In the sphere

$$S_n(K)=\{(x_1,\cdots,x_{n+1});\ \textstyle\sum x_i{}^2=1/K\}$$

of curvature K, consider a submanifold

$$S_{p-1}(K_I)\times S_{n-p}(K_A)=\left\{(x_1,\cdots,x_{n+1});\ \sum_1^p x_i{}^2=1/K_I,\ \sum_{p+1}^{n+1}x_i{}^2=1/K_A\right\},$$

where $1/K_I+1/K_A=1/K$. Then the main theorem in ch. III states that if a V_{n-1} imbedded in $S_n(K)$ is locally reducible and complete, then $V_{n-1}=S_{p-1}(K_I)\times S_{n-p}(K_A)$ up to a motion, $(n\geqq 4,\ p\geqq 2)$. Let $S_n\to P_n$ be the natural projection of the sphere (of curvature K) onto the real projective space. In ch. IV the author proves that if a V_{n-1} in P_n is locally reducible and complete, then V_{n-1} is the image of $S_{p-1}\times S_{n-p}$ under the map $S_n\to P_n$. A non-complete, locally reducible $V_{n-1}{}'$ in S_n (or P_n) is also studied and is proved to be extendable to a complete reducible (or locally reducible) V_{n-1} in S_n (or P_n).

S. Kobayashi (Cambridge, Mass.)

3010:

Nakae, Tatuo. Curvature and relative Betti numbers. J. Math. Soc. Japan **9** (1957), 367–373.

Let M be an n-dimensional orientable Riemannian manifold of class C^∞ with positive definite metric $ds^2=g_{\nu\mu}d\xi^\nu d\xi^\mu$. Let D be an open set in M with $(n-1)$-dimensional boundary B of class C^∞, and $\tilde{D}$ be an open set containing D and B, and assume that $D\cup B$ is compact. The local coordinates in $\tilde{D}$ and B are respectively denoted by ξ^κ $(\kappa,\lambda,\mu,\cdots=1,2,\cdots,n)$ and η^h $(h,i,j,\cdots=1,2,\cdots,n-1)$. We put $B_i{}^\kappa=\partial\xi^\kappa/\partial\eta^i$. Let N^κ be the components of the unit contravariant outward normal vector to the boundary B such that $|N^\kappa, B_1{}^\kappa,\cdots,B_{n-1}{}^\kappa|>0$. The author proves first the formula, valid for a harmonic tensor $v_{\lambda_1\lambda_2\cdots\lambda_p}$ normal to B,

$$\int_D\left[F(v,v)+\frac{1}{p}(\nabla^\mu v^{\lambda_1\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda_1\lambda_2\cdots\lambda_p})\right]d\sigma=\int_B[H_a{}^{a'}v_{i_1\cdots i_p}{}'v^{i_1\cdots i_p}-(p-1)H_{ji}{}'v^j{}_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p}]d'\sigma,$$

where

$$F(v,v)=K_{\mu\lambda}v^{\mu}{}_{\lambda_2\cdots\lambda_p}v^{\lambda\lambda_2\cdots\lambda_p}+\frac{p-1}{2}K_{\nu\mu\lambda\kappa}v^{\nu\mu}{}_{\lambda_3\cdots\lambda_p}v^{\lambda\kappa\lambda_3\cdots\lambda_p}$$

and

$${}'v_{i_2\cdots i_p}=v_{\lambda_1\lambda_2\cdots\lambda_p}N^{\lambda_1}B_{i_2}{}^{\lambda_2}\cdots B_{i_p}{}^{\lambda_p},$$

$K_{\mu\lambda}$ and $K_{\nu\mu\lambda\kappa}$ being Ricci tensor and curvature tensor of D respectively and H_{ji} the second fundamental tensor of B.

From this formula he deduces the theorem. If the quadratic form $F(v,v)$ is positive definite in D and

$$H_a{}^{a'}v_{i_2\cdots i_p}{}'v^{i_2\cdots i_p}-(p-1)H_{ji}{}'v^j_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p}$$

is negative definite on B, then the pth relative Betti number of D is zero for $n-1\geqq p\geqq 1$.

He also proves the formula, valid for a harmonic tensor $v_{\lambda_1\lambda_2\cdots\lambda_p}$ tangential to B,

$$\int_D\left[F(v,v)+\frac{1}{p}(\nabla^\mu v^{\lambda_1\lambda_2\cdots\lambda_p})(\nabla_\mu v_{\lambda_1\lambda_2\cdots\lambda_p})\right]d\sigma=\int_B H_{ji}{}'v^j_{i_2\cdots i_p}{}'v^{ii_2\cdots i_p}d'\sigma,$$

from which he deduces the theorem: If the quadratic form $F(v,v)$ is positive definite and $H_{ji}d\eta^j d\eta^i$ is negative definite, then the pth absolute Betti number of D is zero for $n-1\geqq p\geqq 1$. *K. Yano* (Tokyo)

3011:

Hsiung, Chuan-Chih. Curvature and Betti numbers of compact Riemannian manifolds with boundary. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 95–131.

The author generalizes some of the theorems of Bochner and Yano to finite manifolds with boundary. He proves the following theorem:

On a compact orientable manifold M^n with boundary B^{n-1} and positive definite Ricci curvature everywhere, if

$$\frac{n-2p}{n-2}L+\frac{p-1}{(n-1)(n-2)}R>\frac{p-1}{2}C,$$

where

$$C=\sup_\xi\frac{|C_{ijkl}\xi^{ij}\xi^{kl}|}{\xi^{ij}\xi_{ij}},$$

C_{jkl}^{i} is Weyl's conformal curvature tensor, and L is the smallest (positive) eigenvalue of the matrix $\|R_{ji}\|$; then for $p=1,2,\cdots,[n/2]$ there exists no skew-symmetric harmonic tensor field of order p, other than the zero tensor field, with zero tangential or normal component on the boundary B^{n-1}, and for $p=1,2,\cdots,n-1$ the absolute pth Betti number of M^n is zero.

J. J. Kohn (Waltham, Mass.)

3012:

Yano, Kentaro. Harmonic and Killing tensor fields in Riemannian spaces with boundary. J. Math. Soc. Japan **10** (1958), 430–437.

In this paper the author generalizes his previous results on harmonic and Killing vector fields on Riemannian manifolds with boundary to tensor fields.

Let M be a finite Riemannian manifold with boundary B. Denote by $K_{\mu\lambda}$ the Ricci tensor and by $K_{\nu\mu\lambda\kappa}$ the curvature tensor, and let

$$F(v, v) = K_{\mu\lambda}v^{\mu}_{\lambda_2}\cdots_{\lambda_p}v^{\lambda\lambda_2\cdots\lambda_p} + \frac{p-1}{2}K_{\nu\mu\lambda\kappa}v^{\nu\mu}_{\lambda_3}\cdots_{\lambda_p}v^{\lambda\kappa\lambda_3\cdots\lambda_p}.$$

Then the author proves the following: Theorem. If the quadratic form $F(v, v)$ is positive [negative] definite and the second fundamental form is negative [positive] semi-definite, then there does not exist a harmonic [Killing] tensor field of order p $(1 \leq p \leq n-1)$ tangential to B other than the zero tensor.

The author also proves a dual to this theorem and several theorems giving necessary and sufficient conditions for a tensor field to be harmonic or Killing.

J. J. Kohn (Waltham, Mass.)

3013:

Karpelevič, F. I. Geodesics and harmonic functions on symmetric spaces. Dokl. Akad. Nauk SSSR **124** (1959), 1199–1202. (Russian)

The author considers the geodesics on a homogeneous, symmetric space $M = G/K$, where G is a connected semi-simple Lie group and K its maximal compact subgroup: such a space has defined on it a Riemannian metric whose curvature is constant under parallel displacement and non-positive on all plane elements. Let γ be a geodesic relative to this metric, x an arbitrary point. There is a unique geodesic arc from x perpendicular to γ; let $\rho(x, y)$ be the length of this arc. If γ_0 is another (directed) geodesic, then let $\rho(\gamma_0, \gamma) = \lim_{x\to\infty,\, x\in\gamma_0}\rho(x, \gamma)$. This defines a metric on the set of directed geodesics, except that $\rho(\gamma_0, \gamma) = 0$ does not imply $\gamma = \gamma_0$, nor is the distance always finite. A null path is the set of all γ such that $\rho(\gamma_0, \gamma) = 0$. The null paths then form a metric space P (same metric), and if Γ is a null path, let $P(\Gamma)$ be the set of all null paths at finite distance from Γ. Finally, pass to a decomposition space $M(\Gamma)$ of $P(\Gamma)$ in which are identified all null paths Γ_1, Γ_2 such that $G^0(\Gamma_1) = G^0(\Gamma_2)$, $G^0(\Gamma)$ being the subgroup of all $g \in G$ such that $\rho(g\gamma, \gamma) = 0$, $\gamma \in \Gamma$ ($G^0(\Gamma)$ is independent of the $\gamma \in \Gamma$ chosen); then a theorem of the author states that $M(\Gamma)$ is a symmetric space of the same type as M.

Then the author defines a class of harmonic functions on M which can be considered as continuous functions of the null paths. Further theorems (1) state that these functions are the same on Γ_1, Γ_2 if $G^0(\Gamma_1) = G^0(\Gamma_2)$, hence define functions on $M(\Gamma)$; (2) give an integral formula for the functions (analogous to the Poisson integral formula); and (3) show that if $x_1 \neq x_2$ are arbitrary points, Γ an arbitrary null path, then there exist harmonic functions f_1, f_2 such that $f_1(x_1) \neq f_1(x_2)$ and $f_2(x_1) \neq f_2(\Gamma)$. This last leads to a compactification of M.

In the case of bounded homogeneous domains in complex space the latter results are related to those of D. Lowdenslager [#2836 above].

W. M. Boothby (St. Louis, Mo.)

3014:

Berezin, F. A. An analog of Liouville's theorem for symmetric spaces of negative curvature. Dokl. Akad. Nauk SSSR **125** (1959), 1187–1189. (Russian)

In the quotient space of a semisimple Lie group over its maximal compact subgroup every bounded function satisfying the Laplace equation also annihilates the generalized Laplace operators.

H. Freudenthal (Utrecht)

3015:

Auslander, L.; and Markus, L. Classical differential equations on manifolds. Trans. Amer. Math. Soc. **91** (1959), 113–128.

This paper begins by characterizing the differentiable manifolds on which it is possible to form various classical differential equations or systems. When the authors speak of (say) the potential equation they do not mean (originally) the Laplace-Beltrami equation on a Riemannian manifold; rather, they mean that it is possible to cover the differentiable manifold with coordinate neighborhoods so that in each neighborhood the equation is given by

$$\partial^2 u/\partial x_1^2 + \cdots + \partial^2 u/\partial x_n^2 = 0,$$

preserved under coordinate transformations. For dimension $n \geq 3$ they show that the existence of such a covering is equivalent to a realizable reduction of the bundle of bases $B(M^n)$ to a totally disconnected subgroup of $C(n) \times O(n)$. For the operator itself $C(n) \times O(n)$ is replaced by $O(n)$, $n \geq 2$. In this case M^n admits a flat Riemannian tensor so that the potential operator can be viewed as the Laplace-Beltrami operator for special (flat) tensors. Similar results are obtained for other equations.

Using a holonomy transformation an element of the 1-dimensional real singular cohomology group of an affine manifold M^n is determined which is called the unimodular obstruction of the affine coordinate covering on M^n. This is used to investigate the question of when a manifold which admits the potential equation also admits the potential operator.

For a compact manifold with a flat Riemann metric the eigenfunctions and eigenvalues of the Laplace operator are studied in some detail, using the flat torus T^n which finitely covers M^n. The finite linear combinations of the eigenfunctions form an algebra, which considered as a commutative ring with unit can be embedded in a field of quotients called the eigenfunction field F_M for M^n. The corresponding field F_T for T^n contains F_M. The transcendence degree of F_M over the reals in n. F_T is a finite, normal, algebraic extension of F_M, and the Galois group of F_T over F_M is isomorphic to the holonomy group $H(\Gamma, M^n)$, where Γ is the flat affine connection.

If T^n has edges of length 2π its eigenvalues are non-positive integers, as are those of the covered manifold. For $n \geq 4$ every negative integer occurs for the flat torus (sides 2π).

The wave equation is studied for flat Lorentz tensors. Detailed examples are given of linear differential equations with constant coefficients.

M. P. Gaffney (Washington, D.C.)

3016:

Saeki, Takuya. The holonomy covering space in principal fibre bundles. Tôhoku Math. J. (2) **10** (1958), 304–312.

The author considers a locally flat connection (zero curvature) in a differentiable principal fiber bundle. The holonomy covering space is defined as in L. Auslander and L. Markus, Ann. of Math. (2) **62** (1955), 139–151 [MR **17**, 298], that is, the minimal covering space for which the lifted connection has zero holonomy. It is shown that the holonomy covering space is essentially a universal covering space with regard to the holonomy group. Similar results appear in the paper reviewed above.

L. Markus (Minneapolis, Minn.)

3017:

Takizawa, Seizi. On Cartan connexions and their torsions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 199–217.

The goal of this paper is to give a definition in the language of fibre bundles of the notion of Cartan connection and, after characterizing this type of connection, to discuss some of its properties. Cartan connections were first discussed from this point of view by C. Ehresmann [Colloque de topologie (espaces fibrés), Bruxelles (1950), pp. 29–55, Thone, Liege; Masson, Paris; 1951; MR **13**, 159] and the author, of course, builds on this foundation. A similar, independent clarification of Cartan connections appears in the work of S. Kobayashi [Canad. J. Math. **8** (1956), 145–156; MR **17**, 1926].

A brief sketch of the results is as follows: Let M be a differentiable manifold, $T(M)$ its tangent bundle, and B a bundle over it with structure group $\hat{G}$ (a Lie group) and fibre F. If $\hat{G}$ acts transitively on F with isotropy group G, if there is a cross section $f: M \to B$, and if there is a (linear) identification $i: T(M) \to \mathfrak{T}(M)$, where $\mathfrak{T}(M)$ consists of the submanifold of $T(B)$ of all vectors tangent to the fibre Fx at $f(x)$ for each $x \in M$, then we say the bundle B is "soldered" to M. The existence of f implies that the group of B can be reduced to G. Let B^0 be the principal bundle with group and fibre G which is so determined. Then the existence of the identification i determines canonically a tensorial 1-form ω_0 on B^0 whose vanishing on a vector implies it is vertical, and conversely, such a form determines a soldering. Now suppose a 1-form $\hat{\omega}$ determines a connection on $\hat{B}$, the principal bundle with group $\hat{G}$ associated with B. It is a Cartan connection of the soldered structure if its restriction to B^0 (considered as a submanifold of $\hat{B}$) coincides with $p^*\omega_0$, where $p: \hat{G} \to \hat{G}/G = F$ and p^* is the corresponding map of the Lie algebras. Using these definitions some of the properties of the Cartan connection are then obtained. It is shown that an absolute parallelism can be defined on $T(B^0)$ using the Cartan connection. Formulas for curvature and torsion of the connection are derived and various related results are proved.

W. M. Boothby (St. Louis, Mo.)

3018:

Ōtsuki, Tominosuke. On the tangent bundles of order 2 and affine connections. Proc. Japan Acad. **34** (1958), 325–330.

By using a method similar to the theory of jets of C. Ehresmann [C. R. Acad. Sci. Paris **233** (1951), 598–600; MR **13**, 386] the author defines a second order tangent bundle to a differentiable manifold X. The group L_n^2 of this bundle consists of all pairs $\alpha = (a_i^j, a_{ik}^j)$ with $i, j, k = 1, 2, \cdots, n$ and (a_i^j) non-singular, group multiplication being given by the formulas

$$a_i^j(\alpha\beta) = a_k^j(\alpha)a_i^k(\beta),$$
$$a_{ik}^j(\alpha\beta) = a_h^j(\alpha)a_{ik}^h(\beta) + a_{hl}^j(\alpha)a_i^h(\beta)a_k^l(\beta),$$

where α, β are arbitrary elements of L_n^2. Then using suitable transition functions he defines a bundle $T^2(X)$, the tangent bundle of order 2, which has the usual tangent bundle $T(X)$ imbedded in it and the group of $T(X)$, i.e. $L_n^1 = \mathrm{GL}(n, R)$, as a subgroup of the group L_n^2 of $T^2(X)$. Using $B^2(X)$, $B(X)$ to denote the associated principal bundles of the two tangent bundles one sees that $B^2(X)$ is actually a bundle over $B(X)$. The main result is the theorem: the connections on $T(X)$ are in a natural 1-1 correspondence with the cross-sections ρ of $B^2(x)$ over $B(x)$ which are invariant under right translations, i.e. $\rho\alpha = \alpha'\rho$ where $\alpha \in \mathrm{GL}(n, R)$ and α' is the corresponding element of L_n^2 relative to the natural imbedding.

Further considerations enable the author to derive affine, projective, and conformal connections as special cases.

W. M. Boothby (St. Louis, Mo.)

3019:

Murgescu, Viorel. Observations sur les espaces à torsion, qui admettent une métrique. Bul. Inst. Politehn. Iași (N.S.) **4** (8) (1958), 49–60. (Romanian. Russian and French summaries)

A space with an affine connection is said to be a quasi-recurrent space if its curvature tensor satisfies the relation

$$\nabla_l R_i^h{}_{jk} = \sigma_l R_i^h{}_{jk} + \delta_i^h \tau_{jkl}.$$

It is shown that a quasi-recurrent space admits a scalar $\Lambda(x, v)$, called metric, depending on the pair of a point and a vector, homogeneous in v^i and invariant with respect to parallel displacement if and only if σ_l is a gradient. In the case of a generalized Einstein space ($q_{jl} = \varphi R_{ij}$) a Λ with the above properties exists if and only if the space is Ricci-recurrent. The paper contains further results in connection with generalized Riemann spaces introduced by Eisenhart.

G. Soós (Debrecen)

3020:

Petrescu, Șt. Sur la connexion induite dans un espace non holonome X_n^m. Bul. Inst. Politehn. București **18** (1956), no. 3–4, 57–65. (Romanian. Russian and French summaries)

The author considers a metric space X_n with affine connexion Γ_{jl}^h with torsion and two complementary non-holonomic subspaces X_n^m and X_n^{n-m}. Denoting by E_j^i and F_j^i the projection operators on X_n^m and X_n^{n-m} respectively, he puts

$$M_{jk}^i = \Gamma_{jk}^i - E_{j;k}^i \quad N_{jk}^i = \Gamma_{jk}^i - F_{j;k}^i,$$

the semi-colon denoting the covariant derivative with respect to Γ_{jk}^i.

The M_{jk}^i [N_{jk}^i] defines the so-called induced affine connexion on X_n^m [X_n^{n-m}].

The author gives a necessary and sufficient condition that an affine connexion M_{jk}^i [N_{jk}^i] gives the induced connexion on X_n^m [X_n^{n-m}] in the original space X_n.

K. Yano (Tokyo)

3021:

Petrescu, Şt. Sur les espaces A_2 localement euclidiens. Bul. Inst. Politehn Bucureşti **18** (1956), no. 3–4, 67–80. (Romanian. Russian and French summaries)

An affine connexion in a two-dimensional locally Euclidean space A_2 is given by six functions $\Gamma_{12}{}^1 = \Gamma_{21}{}^1 = u$, $\Gamma_{12}{}^2 = \Gamma_{21}{}^2 = v$, $\Gamma_{11}{}^1 = \alpha$, $\Gamma_{22}{}^2 = \beta$, $\Gamma_{22}{}^1 = a$, $\Gamma_{11}{}^2 = b$. These functions satisfy first order partial differential equations given by the vanishing of the curvature tensor. The author gives various conditions on u, v, α, β, a and b and solves the partial differential equations in each case. *K. Yano* (Tokyo)

3022:

Bark, O. On the projective connection space and general projective geometry of paths. Kyungpook Math. J. **1** (1958), 1–12.

This is an expository article on the projective geometry of paths. Some parts of the classical theory due to J. Douglas [Ann. of Math. (2) **29** (1928), 143–168], L. Berwald [ibid. **37** (1936), 879–898] and K. Yano [Proc. Phys.-Math. Soc. Japan (3) **24** (1942), 9–25; MR **7**, 330] are reconsidered from the point of view developed by E. Cartan [*Leçons sur la théorie des espaces à connexion projective*, Gauthier-Villars, Paris, 1937]. *E. T. Davies* (Southampton)

3023:

Ghosh, N. N. Studies in contact transformation over a complex space. Bull. Calcutta Math. Soc. **50** (1958), 94–98.

The author considers, in the spirit of his paper in same Bull., **42** (1950), 65–72 [MR **12**, 444] contact transformations in real n-space, using the complex space defined with the coordinates $x^r = q^r + ip^r$, $r, s, k, l = 1, \cdots, n$, and their linear transformations. A covariant tensor $C_{\mu\nu}$ of $4n^2$ components is formed: $C_{rs} = 0$, $C_{rs^*} = i = \sqrt{(-1)}$ $(r = s)$ and 0 $(r \neq s)$, $C_{r^*s^*} = \text{conj. } C_{rs}$, $C_{r^*s} = \text{conj. } C_{rs^*}$. The metric tensor $g_{kl} = g_{l^*k^*} = \text{conj. } g_{k^*l^*}$, $g_{kl^*} = g_{lk^*} = \text{conj. } g_{lk^*}$ satisfies the relation $C^{\mu\alpha} g_{\nu\lambda^*} = C_{\mu\nu} g^{\mu^*\alpha}$, and the identity $C_{\mu\nu;\alpha} = 0$ holds; here the covariant differentiation is framed with $\{{}_\mu{}^\sigma{}_\nu\} = \frac{1}{2} g^{\sigma\lambda^*}(\xi_\mu g_{\lambda\nu} + \xi_\nu g_{\mu\lambda} - \xi_\lambda g_{\mu\nu})$, where $\xi = \frac{1}{2}(\partial/\partial q + i\partial/\partial p)$. The Hamilton-Jacobi equations appear as $C_{\mu\nu} dx^{\nu^*}/dt = \xi_\mu(2H)$. *D. J. Struik* (Cambridge, Mass.)

3024:

Ahn, Jae Koo. On the conformal correspondence between the first and second parameter-group spaces. Kyungpook Math. J. **1** (1958), 85–88.

Given a Lie group, consider an inner product on the tangent space at the identity. It induces two Riemannian metrics on the group space; one by the left translation and the other by the right translation. The author gives, in terms of structure constants, a necessary and sufficient condition for these two metrics to be conformally equivalent. *S. Kobayashi* (Cambridge, Mass.)

3025:

Nagai, Tamao. Some properties of subgroups of simply transitive groups. Tensor (N.S.) **7** (1957), 103–109.

In an earlier paper [Tensor (N.S.) **5** (1955), 91–94; MR **17**, 1044] the author studied a simply transitive group G_m and an associated Riemannian manifold R_m. Here a subgroup G_n and an associated subspace R_n are also studied. Using the symbols of T. K. Pan for the variation and subvariation of one congruence of an orthogonal ennuple with respect to another [Canad. J. Math. **5** (1953), 519–523; MR **15**, 350], the author presents simple proofs for eight theorems which deal, among other things, with R_n as a minimal variety of R_m; R_n as a totally geodesic subspace of R_m; a necessary and sufficient condition that G_n be Abelian in the case where G_m has skew-symmetric structure constants; a necessary and sufficient condition that G_n be an invariant subgroup in the case where the vectors that describe G_m are orthogonal normal congruences in R_m; and also a necessary and sufficient condition that G_m be the direct product of G_n and G_{m-n} in this latter case. *A. Schwartz* (New York, N.Y.)

3026:

Mutō, Yosio. On some properties of a kind of affinely connected manifolds admitting a group of affine motions. II. Sci. Rep. Yokohama Nat. Univ. Sect. I **6** (1957), 1–13.

In a previous paper [Tensor (N.S.) **5** (1955), 127–142; MR **18**, 67] the author proved that if an n-dimensional manifold A_n with symmetric affine connexion admits a group of affine motions of order $r > n^2 - 2n$ and if $n \geqq 7$, then the curvature tensor of the manifold A_n must be of the form

$$(1) \qquad R_{\nu\mu\lambda}{}^\kappa = B_{\nu\mu\lambda} A^\kappa - A_\nu{}^\kappa P_{\mu\lambda} + A_\mu{}^\kappa P_{\nu\lambda} + (P_{\nu\mu} - P_{\mu\lambda}) A_\lambda{}^\kappa$$

or of the form

$$(2) \qquad R_{\nu\mu\lambda}{}^\kappa = (Q_\nu P_\mu - Q_\mu P_\nu)(P_\lambda A^\kappa + Q_\lambda C^\kappa) - A_\nu{}^\kappa P_{\mu\lambda} + A_\mu{}^\kappa P_{\nu\lambda} + (P_{\nu\mu} - P_{\mu\nu}) A_\lambda{}^\kappa,$$

the latter case being possible only for $n = 7$. If the curvature tensor is of the form (2), then the order r of the group of affine motions which the manifold admits satisfies the inequality $r \leqq n^2 - 3n + 8$.

In the present paper, the author proves the following two theorems: Theorem 1. If an n-dimensional manifold A_n with symmetric affine connexion whose curvature tensor $R_{\nu\mu\lambda}{}^\kappa$ is of the form (1) admits a group of affine motions of order $r > n^2 - 2n$, and if $n \geqq 7$, then the curvature tensor $R_{\nu\mu\lambda}{}^\kappa$ is of the form

$$(3) \qquad R_{\nu\mu\lambda}{}^\kappa = (Q_\nu P_\mu - Q_\mu P_\nu) P_\lambda A^\kappa - A_\nu{}^\kappa P_{\mu\lambda} + A_\mu{}^\kappa P_{\nu\lambda} + (P_{\nu\mu} - P_{\mu\nu}) A_\lambda{}^\kappa$$

or of the form

$$(4) \qquad R_{\nu\mu\lambda}{}^\kappa = [(Q_\nu P_\mu - Q_\mu P_\nu) P_\lambda + (P_\nu R_\mu - P_\mu R_\nu) Q_\lambda + 2(R_\nu Q_\mu - R_\mu Q_\nu) R_\lambda - (Q_\nu P_\mu - Q_\mu P_\nu) R_\lambda] A^\kappa - A_\nu{}^\kappa P_{\mu\lambda} + A_\mu{}^\kappa P_{\nu\lambda} + (P_{\nu\mu} - P_{\mu\nu}) A_\lambda{}^\kappa,$$

the latter case being possible only for $r \leqq n^2 - 3n + 8$, and hence only for $n = 7$. Theorem 2. A necessary and sufficient condition that an n-dimensional manifold, $n \geqq 7$, with symmetric affine connexion which is not projectively flat, admits a group of affine motions of order $r = n^2 - 2n + 5$, is that the curvature tensor of the manifold is of the form (5) $R_{\nu\mu\lambda}{}^\kappa = (Q_\nu P_\mu - Q_\mu P_\nu) P_\lambda A^\kappa$, where the vectors Q_λ, P_λ and A^κ are covariantly constant and satisfy $Q_\lambda A^\lambda = 0$, $P_\lambda A^\lambda = 0$. *K. Yano* (Tokyo)

3027:

Wu, Wen-tsün. On the isotopy of C^r-manifolds of dimension n in euclidean $(2n+1)$-space. Sci. Record (N.S.) **2** (1958), 271–275.

Whitney [Ann. of Math. (2) **37** (1936), 645–680] has proved the theorem that any two C^r-imbeddings of a compact C^r-manifold M of dimension n in R^{2n+2} are regularly C^r-isotopic. The author proves the theorem for an n-manifold in R^{2n+1}, $n>1$. To prove his theorem the author first shows that two such C^r-imbeddings are regularly homotopic with isolated double points of self-intersection at isolated times. These double points are then eliminated following methods of Whitney [ibid. **45** (1944), 220–240; MR **5**, 273].

S. Smale (Princeton, N.J.)

3028:

Dobrušina, I. S. Typical irregularities in mapping a k-dimensional differential manifold into a $(2k-2)$-dimensional vector space. Mat. Sb. N.S. **45 (87)** (1958), 333–368. (Russian)

The author studies the singularities of differentiable maps of a k-manifold M^k into Euclidean $(2k-2)$-space A^{2k-2}. The case $k=2$ has already been treated by H. Whitney [Ann. of Math. (2) **62** (1955), 374–410; MR **17**, 518] but the author apparently is not familiar with this paper.

She considers a map f: $M^k \to A^{2k-2}$ of class $m \geqq 4$, a singular point $a \in M^k$ of f and a local coordinate system $x^1, \cdots, x^k$ about a such that $\partial f(a)/\partial x^1 = 0$. Then a is said to be non-degenerate if for some $i = 2, \cdots, k$ the set of $2k-2$ vectors

$$\frac{\partial^2 f(a)}{(\partial x^1)^2}, \cdots, \frac{\partial^2 f(a)}{\partial x^1 \partial x^{i-1}}, \frac{\partial^2 f(a)}{\partial x^1 \partial x^{i+1}}, \cdots, \frac{\partial^2 f(a)}{\partial x^1 \partial x^k},$$

$$\frac{\partial f(a)}{\partial x^2}, \cdots, \frac{\partial f(a)}{\partial x^k}$$

in linearly independent. (This is for $k>2$; the case $k=2$ is treated separately.) The basic result is that arbitrarily close to any map f of class $\geqq 4$ there is a map g all of whose singular points are non-degenerate. The set of all such non-degenerate maps of class m is everywhere dense in the space of all maps of class m. If a is a non-degenerate singular point of f, then for any neighborhood U of a and any map g sufficiently close to f, there is a singular point of g in U. The set of all singular points of a non-degenerate map f of class m forms a 1-dimensional manifold of class $m-1$.

The author gives canonical forms for f in the neighborhood of a non-degenerate singular point.

There is of course considerable overlap with the work of Whitney [Sympos. Internac. de Topología Algebraica, pp. 285–301, Univ. Nac. Autónoma México and UNESCO, Mexico City, 1958; MR **20** #4877] and Thom [Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87; MR **19**, 310].

N. Stein (New York, N.Y.)

3029:

Baudoin-Gohier, Simone. Rigidités des surfaces convexes à bords. Ann. Sci. École Norm. Sup. (3) **75** (1958), 167–199.

The paper contains a general exposition of results on rigidity of surfaces which derive from the method of Herglotz [Abh. Math. Sem. Hansischen Univ. **15** (1943), 127–129; MR **7**, 322] and its modifications mainly due to Rembs [Math. Z. **56** (1952), 271–279; S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 315–320; MR **14**, 901; **17**, 76] and Grotemeyer [Math. Z. **55** (1952), 253–268; **58** (1953), 41–45; **59** (1953), 278–289; MR **13**, 984; **15**, 57, 462]. The author adds new proofs of known theorems and some new results among which we state as samples: (1) Any open convex surface of class C^3 and everywhere positive Gaussian curvature with the boundary tangent to a cylinder of revolution, possesses global rigidity (G.R.); (2) any open convex surface of class C^3 with the boundary tangent to a sphere possesses G.R.; (3) any open convex surface of class C^2 and total curvature equal to 4π possesses G.R.

L. A. Santaló (Buenos Aires)

3030:

Vilhelm, Václav. Kurven in Minkowskischen Räumen. Časopis Pěst. Mat. **82** (1957), 283–300. (Czech. Russian and German summaries)

Der metrische Tensor g_{ij} eines Minkowskischen n-dimensionalen Raumes M_n ist in jedem seiner Punkte (x) bekannt, wenn dort zusätzlich eine Richtung $\dot{x}^1 : \dot{x}^2 : \cdots : \dot{x}^n$ ausgezeichnet wird. Verfasser setzt $g_{ik}(\dot{x}^1, \cdots, \dot{x}^n)$ nach jedem nichtverschwindenden Vektor $\dot{x}$ stetig differenzierbar voraus und beschränkt sich auf solche Vektoren $\dot{x}$, für welche $|g_{ij}(\dot{x})| \neq 0$. Das skalare Produkt zweier Vektoren a, b in der Richtung c wird durch

$$(a, b)_c = g_{ij}|c^1, \cdots, c^n| a^i b^j$$

definiert. Für $(d, e)_d$ schreibt Verfasser kurz (d, e), wenn $d \neq 0$. Auf diese Weise läßt sich im Minkowskischen Raum eine orthonormale Basis $e_1, \cdots, e_n$ einführen, welche den Bedingungen $(e_i, e_j) = \delta_j^i$ für $i \leqq j$, $j = 1, 2, \cdots, n$ genügt. Ist nun s der Bogenparameter einer in keinem linearen Unterraum des M_n gelegenen kurve k mit den Krümmungen $\varkappa_1(s), \cdots, \varkappa_{n-1}(s)$ und der Tangente $e_1(s)$ und der i-ten Normalen $e_{i+1}(s)$ $(1 \leqq i < n)$, so gilt für die Ableitungen $e_i'(s)$ des begleitenden orthonormalen n-Beins in Analogie zu den Frenetschen Formeln der Kurventheorie im n-dimensionalen euklidischen Raum

$$e_i' = -\alpha_{i1} e_1 - \cdots - \alpha_{ii} e_i + \varkappa_i e_{i+1} \quad (1 \leqq i \leqq n)$$

mit $\varkappa_{n+1} = 0$ und $e_{n+1} = 0$,

$$\alpha_{i1} = \varkappa_1 \cdot (e_2, e_1)_{e_1},$$

$$\alpha_{i1}(e_2, e_1) + \alpha_{i2} = -\alpha_{i1}(e_1, e_i)_{e_2} + \varkappa_2 (e_2, e_i)_{e_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{j=1}^{i-1} \alpha_{ij}(e_{i-1}, e_j) = -\sum_{j=1}^{i-1} \alpha_{i-1,j}(e_j, e_i)_{e_{i-1}} + \varkappa_{i-1}(e_i, e_i)_{e_{i-1}},$$

$$\alpha_{i1}(e_i, e_1) + \cdots + \alpha_{ii} = 0.$$

Damit ergibt sich: zwei orientierte Kurven k_1, k_2 in M_n sind genau dann isometrisch, wenn sie gleiche Bogenlänge, überall gleiche Krümmungen und im Anfangspunkt isometrische begleitende n-Beine haben. Umgekehrt gehört zu jedem System positiv stetiger Funktionen $\varkappa_1(s), \cdots, \varkappa_{n-1}(s)$ mit $0 \leqq s \leqq s_1$ und zu jeder orthonormalen Basis eine orientierte Kurve in M_n mit dem Bogen s, der i-ten Krümmung $\varkappa_i(s)$ und dieser Basis als begleitendes n-Bein im Punkt $s=0$. Verfasser gewinnt auch eine geometrische Deutung der Krümmungen $\varkappa_i(s)$ mit Hilfe des i-ten oskulierenden Raumes der orientierten Kurve k im $k(0)$.

M. Pinl (Köln)

3031:

Ku, Chao-hao. On Finsler spaces admitting a group of motions of the greatest order. Sci. Record (N.S.) **1** (1957), 215–218.

The methods developed by K. Yano [Trans. Amer. Math. Soc. **74** (1953), 260–279; MR **14**, 688] for the study of n-dimensional Riemannian spaces admitting a group of motions of order $\frac{1}{2}n(n-1)+1$ are extended and applied by the author to n-dimensional Finsler spaces, this extension being suggested by the results of H. C. Wang [J. London Math. Soc. **22** (1947), 5–9; MR **9**, 206]. In a Finsler space admitting a group of motions of order $\frac{1}{2}n(n-1)+1$ there exists a unique field of unit vectors which is invariant under the group, by means of which an associate {Rev.: "osculating"} Riemannian metric is defined. This is used to determine the group of motions and the general forms of the fundamental function of the space. *H. Rund* (Durban)

3032:

Borisov, Yu. F. The parallel translation on a smooth surface. I. Vestnik Leningrad. Univ. **13** (1958), no. 7, 160–171. (Russian. English summary)

A method of parallel displacement in the sense of Levi-Civita along a curve L in a surface F is developed, valid for any compact set of rectifiable curves. If $M_0=A, A_1, A_2, \cdots, A_n=B$ are points on L, with tangent planes P_i at A_i, and $\bar{a}_{i+1}$ is the projection of the vector $\bar{a}_i$ at P_i on P_{i+1}, then $\bar{a}_n=\bar{a}(\bar{A})$ is the result of the process of projecting $\bar{a}_0$ through the A_i on P_n. We obtain the parallel translation of $\bar{a}_0$ along L when the points A are infinitely condensed along L. This is possible (with preservation of the length of the vector) when and only when $\lim \sum_i \theta_i^2=0$, where θ_i is the angle between the normals to F at A_{i-1} and A_i. This relates the curve to its spherical image, and the convergence of the limit process is carefully analyzed. *D. J. Struik* (Cambridge, Mass.)

3033:

Borisov, Yu. F. The parallel translation on a smooth surface. II. Vestnik Leningrad. Univ. **13** (1958), no. 19, 45–54. (Russian. English summary)

Parallel translation of a vector, as discussed in the first paper, is investigated if the process converges uniformly on any set of curves of limited length lying in a region with compact closure. This condition is equivalent with the uniform vanishing of $\theta^2(M, N)/\rho_F(M, N)$ for $\rho_F(M, N) \to 0$, where θ is the angle between the normals at M and N, ρ_F is the intrinsic metric of the surface F; M and N are chosen as points in the compact region. Every geodesic line (defined as the shortest in the neighborhood of all its points) is a smooth curve (this concept is precisely defined) and its tangential vector is subjected to parallel translation along it. *D. J. Struik* (Cambridge, Mass.)

3034:

Borisov, Yu. F. The parallel translation on a smooth surface. III. Vestnik Leningrad. Univ. **14** (1959), no. 1, 34–50. (Russian. English summary)

The surfaces for which the condition of article II holds are further investigated. The intrinsic character of parallel displacement is proved, geodesic curvature and the Gauss-Bonnet theorem restated, as well as the determination of a curve on the surface by means of its natural equation: given a certain additive function of intervals there exists a curve with the integral of the geodesic curvature equal to this function. *D. J. Struik* (Cambridge, Mass.)

3035:

Coz, Marcel. Pyramides et cônes convexes asymétriques. C. R. Acad. Sci. Paris **248** (1959), 3682–3684.

PROBABILITY

See also 2599, 2919, 3084, 3085, 3086, 3131.

3036:

★**Derman, Cyrus; and Klein, Morton. Probability and statistical inference for engineers: A first course.** University Texts in the Mathematical Sciences. Oxford University Press, New York, 1959. xii+144 pp. $3.75.

This is a lucidly written introductory book on probability and statistical decision for engineering students who have taken undergraduate courses in integral and differential calculus. The frequency interpretation of probability is stressed and the treatment of conditional probability is not obscured by any reference to Bayes' rule. The section on point estimation is concentrated to a presentation of essentials (unbiasedness, consistency, minimum variance) and, although the minimax and Bayes principles of choice are treated before hypothesis testing and interval estimation, the usefulness of the latter topics in situations where a loss or gain structure is not available is pointed out. Most of the examples and exercises are chosen from engineering and economics.

Contents: Elements of set theory. Functions. Sum and product notations. The sample space. Random variables. Probability. Distribution functions. Distributions of special interest. Distributions of functions of random variables. Some descriptive properties of distributions. Joint distribution functions. Conditional probability. Statistical independence. The binomial distribution. Independent random variables. The law of large numbers. The central limit theorem. Approximations to distributions. Decisions with known distributions. Decisions with unknown distributions. Estimation. Decision rules. Principles of choice. Tests of hypotheses. Confidence interval estimation. Tables of the normal, t, and chi-square distributions. *D. M. Sandelius* (Göteborg)

3037:

★**Carnap, Rudolf; und Stegmüller, Wolfgang. Induktive Logik und Wahrscheinlichkeit.** Springer-Verlag, Vienna, 1959. viii+261 pp. $7.60.

Dies Buch ist im wesentlichen eine von Stegmüller besorgte Übersetzung einer Auswahl aus Carnap: *Logical foundations of probability* [Univ. of Chicago Press, Chicago, Ill., 1950; MR **12**, 664] und: *The continuum of inductive methods* [Univ. of Chicago Press, Chicago, Ill., 1952; MR **14**, 4]. Stegmüller hat ausserdem eine Einleitung hinzugefügt. Durch die Reduktion der sehr breit geschriebenen "foundations" auf ca. 200 Seiten ist ein sich besser lesbares Buch entstanden—so dass der Ref. fast

eine Rückübersetzung empfehlen möchte. In zwei Anhängen wird über Carnaps "Bedeutungspostulate" und über seine Axiomatik der komparativen Wahrscheinlichkeit berichtet. Neu ist nur ein vereinfachtes Axiomensystem für die c-Funktionen, bei dem auch der Fall berücksichtigt wird, dass die Grundsprache k Prädikate ($k>2$) enthält, die eine vollständige Disjunktion bilden.

P. Lorenzen (Kiel)

3038:

Pratt, John W. On a general concept of "in probability". Ann. Math. Statist. **30** (1959), 549–558.

The author refers to a paper of Mann and Wald in which the O and o notation is extended to statements in probability, O_p and o_p, and in which it is shown that the calculus of O and o also extends. The proofs and some applications of their results make implicit use of the equivalences among (1) $X_n/r_n \to 0$ in probability, (2) For every positive ε, there exist c_n with $P_n\{|X_n| \leq c_n\} \geq 1-\varepsilon$ such that $c_n = o(r_n)$, (3) For every positive ε, there exist S_n with $P_n\{X_n \in S_n\} \geq 1-\varepsilon$ such that $x_n \in S_n$ for all n implies $x_n = o(r_n)$. The latter definition is extended by the author to define a general concept of "in probability". "$(X_1, X_2, \cdots) \in S$" is said to occur in probability if for every positive ε, $P_n\{X_n \in S_n\} \geq 1-\varepsilon$ and $X_n \in S_n$ for all n implies $(X_1, X_2, \cdots) \in S$. A few fundamental results based on the use of this broader definition seems to remove some of the complexities in the proofs, statements and applications of the Mann-Wald results. Further generalizations are considered.

H. Chernoff (Stanford, Calif.)

3039:

Rényi, Alfréd. On a one-dimensional problem concerning random space filling. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 109–127. (Hungarian. Russian and English summaries)

The author solves the one dimensional analogue of a problem raised for three dimensions by Bernal and Schmetterer. Let us place at random a unit interval in the interval $(0, x)$ (i.e., its center is uniformly distributed in $(\frac{1}{2}, x-\frac{1}{2})$). Then we place a second interval at random independently of the first in such a way that it does not intersect the first. The process is repeated until there is no further possibility of placing another unit interval so that it should not intersect any of the previously placed intervals. Let us denote by v_x the number of these intervals; v_x is clearly a random variable. Denote by $\mathcal{M}(x)$ its mean value. The author proves that for $x \geq 0$

$$(1)\quad \mathcal{M}(x+1) = \frac{2}{x}\int_0^x \mathcal{M}(t)dt + 1, \quad \mathcal{M}(x) = 0 \text{ for } 0 \leq x \leq 1.$$

From (1) he deduces that

$$\lim_{x=\infty} \frac{\mathcal{M}(x)}{x} = \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}\,du\right)dt \sim 0.748.$$

He also estimates the variance of v_x. Several further results are proved. A possible application is about parking of cars.

P. Erdös (Budapest)

3040:

Englman, R. The eigenvalues of a randomly distributed matrix. Nuovo Cimento (10) **10** (1958), 615–621. (Italian summary)

The distribution of the eigenvalues of a symmetric matrix with randomly distributed elements is considered. By resolving the matrix into the product of two triangular matrices, the problem is reduced to finding the solution of certain integral equations; these are developed in some special cases. The results are relevant to obtaining the eigenfrequencies of a disordered linear chain.

S. Simons (London)

3041:

Yang, T. B. Une remarque sur l'esperance conditionnelle. Advancement in Math. **3** (1957), 658–661. (Chinese)

Algebraic version of Shu-Teh chen Moy's characterization of conditional expectation as a transformation on the space of measurable functions [Pacific J. Math. **4** (1954), 47–63; MR **15**, 722].

K. L. Chung (Syracuse, N.Y.)

3042:

Revuz, André. Sur les répartitions de probabilités dans les espaces topologiques ordonnés. J. Math. Pures Appl. (9) **36** (1957), 147–153.

In an earlier paper [Ann. Inst. Fourier, Grenoble **6** (1955–1956), 187–269; MR **19**, 536] the author studied increasing functions and measures on ordered topological spaces (o.t.s.). In this paper the results of the above study are used to construct probability distributions on o.t.s. Let $F(x)$ be a positive, real-valued totally increasing function, continuous on the right, defined on an o.t.s. X. If there exists a minimal element x_0 and a maximal element x_1 in X such that $F(x_0)=0$ and $F(x_1)=1$, then $F(x)$ is a probability distribution on X. Notation: $C_-(x)=\{y: y<x\}$, $C_+(x)=\{y: y>x\}$; $V(x)$ is a neighbourhood of x in the topology T of X, with the sets $V(x)\cap C_+(x)$ forming a base of the filter of neighbourhoods $V_+(x)$ of x for the topology T_+ (here continuity on the right means continuity relative to T_+);

$$\begin{aligned}\Delta(S) = {} & F(x) - \sum_1 F(\inf xu_i) + \cdots \\ & + (-1)^p \sum_p F(\inf xu_{i_1}\cdots u_{i_p}) + \cdots \\ & + (-1)^n F(\inf xu_1 \cdots u_n),\end{aligned}$$

where S is a system of $n+1$ elements $x, u_1, u_2, \cdots, u_n \in X$.

Four theorems are stated and proved. The main results are expressed by the following. Theorem 3: Let U be a subspace of an o.t.s. X satisfying the following conditions: (U_0) U is a semi-lattice for the order induced by that of X; (U_a) for all $x \in U$, $C_-(x)$ is compact; (U_b) $\inf xy$ is continuous on the right on U; (U_c) for all $x \in U$ and all $V_+(x)$, there exists a $y \in U \cap V_+(x)$ such that $C_-(x) \subset C_-^0(y)$ (the interior of $C_-(y)$). Under these conditions, if $F(x)$ is a positive real-valued function defined on U, continuous on the right on U and totally increasing on U, there exists a countably additive measure μ defined on a σ-ring comprising the cones $C_-(x)$, $x \in U$, and such that $\mu C_-(x) = F(x)$. (This theorem generalizes the construction due to A. N. Kolmogorov for the space $\bar{R}^M$.) Theorem 4: If the function $F(x)$ is totally increasing and continuous on the right on a subspace U of X, satisfying U_0, U_a, U_b, U_c and everywhere dense on X for T_+, there exists a unique extension $\bar{F}$ of F on the whole of X that is totally increasing and continuous on the right.

A. T. Bharucha-Reid (Eugene, Ore.)

3043:

Postnikov, A. G. A solution of a set of simultaneous difference equations corresponding to the Dirichlet problem by means of a normal sequence of digits. Dokl. Akad. Nauk SSSR **123** (1958), 407–409. (Russian)

Let S be any normal sequence $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots$ of digits selected from a set G of integers, namely $0, 1, \cdots, g-1$; $g \geqq 2$. Also let $\Delta = (\delta_1, \delta_2, \cdots, \delta_s)$ be any ordered set of S integers δ_i selected from G and define the measure of Δ as $\mu\Delta = 1/g^s$. The measure $\mu\mathfrak{R}$ of a given collection of Δ-sets of various lengths S is defined as the sum of the measures of the individual Δ-sets composing $\mathfrak{R}$.

In this paper a solution of the two-dimension Dirichlet difference equation problem is obtained by using a normal sequence S when $g=4$. To validate the use of S for this purpose, a limit theorem is proved relating the $\mu\mathfrak{R}$ for a collection $\mathfrak{R}$ of Δ-sets of a special type and the number of appearances of these sets in S.

H. L. Turrittin (Minneapolis, Minn.)

3044:

Wang, Shou-jen. A remark on interpolation in a homogeneous random field on the lattice points in R_k. Advancement in Math. **3** (1957), 257–262. (Chinese. English summary)

Generalization of a formula by Kolmogorov [Izv. Akad. Nauk SSSR. Ser. Mat. **5** (1941), 3–14; MR **3**, 4] to high dimensions, but under the assumption of an absolutely continuous spectral function.

K. L. Chung (Syracuse, N.Y.)

3045:

Gaek, Yaroslav [Hájek, Jaroslav]. On a property of normal distribution of any stochastic process. Czechoslovak Math. J. **8** (**83**) (1958), 610–618. (Russian. English summary)

Let $P\{\cdot\}$ and $Q\{\cdot\}$ be two distribution functions defined on a Borel field $\mathfrak{F}$. $P\{\cdot\}$ and $Q\{\cdot\}$ are equivalent if and only if $P\{\Lambda\}=0$ implies $Q\{\Lambda\}=0$ for every $\Lambda \in \mathfrak{F}$; $P\{\cdot\}$ and $Q\{\cdot\}$ are singular if there exists an event $\Lambda_0 \in \mathfrak{F}$ such that $P\{\Lambda_0\}=0$ and $Q\{\Lambda_0\}=1$. Suppose now that $P\{\cdot\}$ and $Q\{\cdot\}$ are two arbitrary normal distributions of a stochastic process $\{x_t, t \in T\}$; it is shown that $P\{\cdot\}$ and $Q\{\cdot\}$ are either equivalent or singular depending upon whether the J-divergence between P and Q is either $< \infty$ or $= \infty$ [see also the author's note published in same J. **8** (**83**) (1958), 460–463; MR **20** #6150]. Finally, two applications are given for the change of the scale parameter and for the translation of a Gauss-Markov process.

R. Theodorescu (Bucharest)

3046:

Korezlioglu, Hayri. Sur la prévision et le filtrage linéaires des processus discrets du second ordre. C. R. Acad. Sci. Paris **248** (1959), 356–357.

The author studies a second order stochastic process x_n with discrete time n. Representing it as

$$x_n = \sum_{j_k \leqq n} c_{n,j_k}\, \xi_{j_k},$$

where the variables ξ_{j_k} form an orthonormal basis of the subspace $L_n(x)$, he shows that

$$\log |c_{n,n}|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |\gamma_n(\lambda)|^2 d\lambda.$$

Here

$$x_n = \int_{-\pi}^{\pi} \gamma_n(\lambda) d\xi(\lambda),$$

with

$$E d\xi(\lambda)\overline{d\xi(\mu)} = \frac{d\lambda}{2\pi}\,\delta_{\lambda,\mu},$$

and the function $\gamma_n(\lambda)$ is constructed in a certain way from the $c_{n,m}$. The construction, a Fourier series expansion, is not given explicitly.

The author also studies linear filtering of two second order processes, using their representations in terms of orthogonal processes.

U. Grenander (Stockholm)

3047:

Urbanik, K. Effective processes in the sense of H. Steinhaus. Studia Math. **17** (1958), 335–348.

If relative (Lebesgue) measure on $[0, \infty)$ and the corresponding relative measurability of functions defined on this interval are defined in the obvious way, a family of jointly relatively measurable functions is a family of random variables, that is, a stochastic process. If f is a function on $[0, \infty)$ whose translation through t units is f_t, f is said to determine 'an effective process with independent increments' if the family of translates $\{f_t, 0 \leq t < \infty\}$ is such a process, in terms of relative measure, and if, whenever $0 \leq a < b$, the increment $f(b+t)-f(a+t)$ determines a relatively measurable function of t, having a distribution function with the usual properties. It is proved that if a stochastic process of the usual type is measurable, separable, and has stationary independent increments, then almost every sample function determines an effective process with independent increments for which the relative measure distribution function of an increment is the corresponding increment distribution function.

J. L. Doob (Urbana, Ill.)

3048:

Korezlioglu, Hayri. Extension du théorème de Pinsker. C. R. Acad. Sci. Paris **248** (1959), 523–525.

Consider two regular Gaussian processes x and y representable as integrals with respect to two orthogonal stochastic set functions ξ and η. The author shows that the average information for one of the processes, knowing the other one, is bounded from below by an expression in $f_{t\eta}(\lambda)$, where

$$f_{t\eta}(\lambda) d\lambda = 2\pi E d\xi(\lambda)\, \overline{d\eta(\lambda)}.$$

This result generalizes a result due to Pinsker on stationary processes.

U. Grenander (Stockholm)

3049:

Sack, R. A. Equivalence of two absorption problems with Markovian transitions and continuous or discrete time parameters. Proc. Cambridge Philos. Soc. **55** (1959), 177–180.

Let $X(t)$, $t \geqq 0$, be a Markov process with a countable state space composed of ν irreducible closed sets $C_1, C_2, \cdots, C_\nu$ and a non-empty transient set of states T. Let $\omega_k = \lim_{n\to\infty} \Pr\{X(t) \in C_k\}$ and suppose $\Pr\{X(0) \in T\} = 1$. The author remarks that there is a related discrete time Markov process Y_n $(n = 0, 1, \cdots)$ such that

$$\omega_k = \lim_{n\to\infty} \Pr\{Y_n = E_k\}.$$

The corresponding remark if the original state space is uncountable may be false.

D. A. Darling (Ann Arbor, Mich.)

3050:

Longo, Antonio. Un criterio di previsione di Wiener nella statistica. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 7–23.

An expository paper on prediction theory of stationary time series. *J. Feldman* (Berkeley, Calif.)

3051:

Spitzer, Frank. Some theorems concerning 2-dimensional Brownian motion. Trans. Amer. Math. Soc. **87** (1958), 187–197.

If $X(t)$ and $Y(t)$ are the co-ordinates, at the time t, of the particle in a two-dimensional Brownian motion, and $R(t)=|X(t)+iY(t)|$, it is proved that, given any positive non-increasing function $g(t)$, $R(t)\geqq g(t)t^{1/2}$ for all sufficiently large t with probability 1 or 0 according as the series $\sum_{k=1}^{\infty}[k|\log g(k)|]^{-1}$ converges or diverges. This is the counterpart of a proposition by A. Dvoretsky and P. Erdös [Proc. 2nd Berkeley Sympos. Math. Statist. and Probabil., 1950, pp. 353–367, Univ. of California Press, Berkeley-Los Angeles, 1951; MR **13**, 852] dealing with Brownian motion in three or more dimensions. A conjecture by P. Lévy [*Le mouvement brownien*, Mémor. Sci. Math., no. 126, Gauthiers-Villars, Paris, 1954; MR **16**, 601; see his p. 59] is thereby disproved.

Furthermore, if $\theta_R(t)=\arg[X(t)+iY(t)]$, where $R=R(0)$, the limiting distribution of $2(\log t)^{-1}\theta(t)$ for $t\to\infty$ is found to be the Cauchy distribution. If absorbing barriers are set at $\theta=0$ and $\theta=\beta$, if $0<\theta_R(0)<\beta$, and if $R>0$, the moment of order δ of the absorption time is finite if, and only if, $2\delta\beta<\pi$.

A final section is devoted to the Cauchy process $C(t)$, defined as the separable process with independent increments $C(t+s)-C(s)$ of probability density $t[\pi(t^2+x^2)]^{-1}$. Assuming that, in the Brownian motion discussed above, $X(0)=x$, $Y(0)=0$, a Cauchy process $C(y)$ is generated by the abscissa of the particle when the ordinate is equal to y for the first time. This representation is applied to the solution of absorption problems for the Cauchy process; if $-1<C(0)=x<1$, the probability that $C(t)\geqq 1$ before $C(t)\leqq -1$ is found to be $\frac{1}{2}+\pi^{-1}\sin^{-1}x$.

S. K. Zaremba (Swansea)

3052:

Jensen, Ernst Lykke. Elementary queue theory. Nordisk Mat. Tidskr. **6** (1958), 137–150. (Norwegian)

Expository paper. *H. Wold* (Uppsala)

3053:

Finch, P. D. Balking in the queueing system GI/M/1. Acta Math. Acad. Sci. Hungar. **10** (1959), 241–247. (Russian summary, unbound insert)

Stationary state probabilities are determined for a single server traffic system with strict queueing (service in order of arrival), maximum capacity $N+1$ (customers arriving when the waiting line is of length N leave without service, and without effect on the input), balking according to length of waiting line, recurrent input and exponential distribution of service time. Recurrent input is shorthand for: intervals between arrival epochs are independently and identically distributed with distribution function $A(x)$; the system states are specified by the number of customers present simultaneously. A customer who on arrival finds the system in state m "balks" (abandons service) with probability $1-b_m$; the balking sequence $[b_m]$ is supposed nonincreasing with $b_0=1$. If $p(x)$ is the generating function for the number of departures (service completions) in an inter-arrival interval ($p(x)=a(b-bx)$, with b the service rate and $a(s)$ the Laplace-Stieltjes transform of $A(x)$), and if

$$(1-p(x))q(x) = (1-x)p(x)r(x)$$

with

$$q(x) = 1+q_1x+q_2x^2+\cdots,$$
$$r(x) = q_1b_N+q_2b_{N-1}x+q_3b_{N-2}x^2+\cdots,$$

it is shown first that the stationary limits of state probabilities at times immediately preceding arrival epochs, P_k for state k, are given by

$$P_k = q_{N+1-k}(1+q_1+\cdots+q_{N+1})^{-1}$$

when they exist, which is when the arrival rate is less than the service rate. The corresponding stationary probabilities P_k^* for arbitrary epochs exist if, in addition, $A(x)$ is not a lattice distribution, when they are given by:

$$P_0^* = 1-\rho(b_0P_0+b_1P_1+\cdots b_NP_N),$$
$$P_k^* = \rho b_{k-1}P_{k-1}, \quad k=1,2\cdots,N+1,$$

with $\rho=a/b$, the occupancy.

Results are exemplified by three special cases: (i) no balking, when they agree with those given by Takacs [Ann. Univ. Sci. Budapest Eötvös. Sect. Math. **1** (1958), 73–82; MR **21** #1649]; (ii) Poisson input, when they agree with results of Haight [Biometrika **44** (1957), 360–369; MR **19**, 692], and (iii) Erlangian E_2 input ($a(s)=\lambda^2(\lambda+s)^{-2}$), which are new results.

There seems to be a slip in the first of equations (10): the range of k should be $1, 2, \cdots, N+1$.

J. Riordan (New York, N.Y.)

3054:

Gnedenko, B. V. On a generalization of Erlang's formulae. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 347–350. (Ukrainian. Russian and English summaries)

"This paper gives a generalization of Erlang's formula to the case when the servicing devices may get out of order and require for their repair an expenditure of the operator's time. The case considered is that of n devices and r operators." (From the author's summary)

J. Wolfowitz (Ithaca, N.Y.)

3055:

Smith, Walter L. On the cumulants of renewal processes. Biometrika **46** (1959), no. 1/2, 1–29.

Let $X_1, X_2, \cdots$ be independent non-negative identically distributed non-lattice random variables; let $S_k=X_1+\cdots+X_k$, and let $N_t=\max\{k|S_k\leqq t\}$, with $N_t=0$ if $X_1>t$. Define $\mathcal{F}$ to be the class of distribution functions G such that, for some integer n, the n-fold convolution of G with

itself has an absolutely continuous component. The primary result is that if distr $\{X_1\} \in \mathcal{S}$, and $EX_1^{n+p+1} < \infty$, $p \geq 0$, then the nth cumulant of N_t is of the form $a_n t + b_n + \lambda(t)/(1+t)^p$, where $\lambda(\cdot)$ is of bounded total variation, is $o(1)$ as $t \to \infty$, and $\lambda(t) - \lambda(t-\alpha) = o(t^{-1})$ as $t \to \infty$ for $\alpha > 0$. The function λ depends on the index n of the cumulant. The rest of the paper is a careful and complete discussion aimed at explicit computation of the coefficients a_n and b_n in terms of the moments of X_1. Many of the lemmas of this paper are of interest in analytic probability theory and the theory of transforms of distribution functions, independently of the problem treated here. *V. E. Beneš* (Murray Hill, N.J.)

3056:

Takács, Lajos. On a generalization of the renewal theory. Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 91–103. (Hungarian and Russian summaries)

Suppose that a machine is working intermittently, and is not working at $t=0$. Let the lengths of the successive idle and working periods be $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$. Suppose further that a renewal process is in operation during the actual working time of the machine, but is suspended when the machine is not working, and let the renewals occur at the intervals $\zeta_1, \zeta_2, \cdots$ of "working" time. Thus the machine is idle from $t=0$ until $t=\xi_1$, when it starts working, and the first renewal will occur at the "clock" time $t=\xi_1+\zeta_1$ if $\zeta_1 < \eta_1$, at $t=\xi_1+\eta_1+\xi_2+(\zeta_1-\eta_1)$ if $\eta_1 < \zeta_1 < \eta_1+\eta_2$, and so on. Let N_t denote the number of renewals which occur in the interval $(0, t]$ of "clock" time. It is assumed that the ξ's, the η's, and the ζ's are severally identically distributed, and that they are all mutually independent. For each of the ξ-, the η-, and the ζ-distributions the author assumes (1) that it has a finite second moment, or (2) that the corresponding distribution function $F(\cdot)$ satisfies $1-F(x) \sim A/x^\gamma$ when $x \to \infty$, where $0 < \gamma < 1$, or (3) $1 < \gamma < 2$. For the resulting 27 possibilities, further refined to 59 possibilities, the author computes the asymptotic distribution of the random variable N_t for large t. In particular when all of ξ, η, and ζ have finite variances he shows that

$$\lim_{t\to\infty} \operatorname{pr}\{N_t - at \leq bt^{1/2}x\} = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2}\, dy,$$

where the constants a and b are given explicitly in terms of the means and variances of ξ, η, and ζ. The principal tool is a theorem of R. L. Dobrušin [Uspehi Mat. Nauk **10** (1955); no. 2 (64), 157–159; MR **17**, 48].

D. G. Kendall (Oxford)

3057:

Ivanenko, I. P. On the equilibrium angular distribution function of particles in a cascade shower. Soviet Physics. Dokl. **122** (3) 1958), 962–966 (367–370 Dokl. Akad. Nauk SSSR).

The model of the electron-photon cascade used by the author is that of Tamm and Belenky, with highly simplified bremsstrahlung and pair production cross-sections, and a constant rate of energy loss through ionisation. Expressions are obtained for the electron angular distribution at shower maximum, as a closed expression for small angles and an infinite sum of Legendre polynomials for large angles. *B. A. Chartres* (Sydney)

STATISTICS

See also 3036.

3058:

Karlin, Samuel; and Rubin, H. Distributions possessing a monotone likelihood ratio. J. Amer. Statist. Assoc. **51** (1956), 637–643.

3059:

St-Pierre, J.; and Zinger, A. The null distribution of the difference between the two largest sample values. Ann. Math. Statist. **27** (1956), 849–851.

3060:

Shimada, Shozo. Moments of order statistic drawn from exponential distribution. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1957), 153–158.

3061:

Darwin, J. H. On corrections to the chi-squared distribution. J. Roy. Statist. Soc. Ser. B. **20** (1958), 387–392.

Karl Pearson's chi-square statistic is considered for a sample of N from a multinomial distribution with k cells. Hoel has shown [Ann. Math. Statist. **9** (1938), 158–165] that if the discreteness of the statistic is ignored, the large sample distribution has a density which differs from that of χ^2 with $k-1$ degrees of freedom by an amount $O(N^{-1})$. It is shown here that the error due to discreteness is $O(N^{-k/4})$. The method involves the Laplace transform and approximations to Bessel functions for arguments of large modulus. *R. F. Tate* (Seattle, Wash.)

3062:

Silvey, S. D. The Lagrangian multiplier test. Ann. Math. Statist. **30** (1959), 389–407.

The work presented here is a continuation of previous work [same Ann. **29** (1958), 813–828; MR **20** #1382]. Basically there is the problem of estimating a finite dimensional parameter under the restriction that its true value be a member of a proper subset of the parameter space. The estimation problem is handled by locating a maximum of the likelihood function by the Lagrange multiplier method. The use of this principle in testing the hypothesis that the true parameter lies in the prescribed subset is discussed in this paper, along with a consideration of the power of such a test.

The author details the assumptions which he imposes to define acceptable properties for this test. Under a specified set of assumptions the asymptotic joint distribution of the parameter estimator and Lagrange multiplier is obtained. Under the same assumptions, the asymptotic distribution of a statistic involving only the multiplier is obtained. In order to discuss power of the test, attention is paid to the situation where the true parameter value does not belong to the set in which the estimate is restricted to lie. A comparison is made between the well-known likelihood ratio test, the Wald test (where acceptance of the hypothesis is based on how well the unrestricted maximum likelihood estimate satisfies the subset-defining conditions) and the Lagrangian multiplier test (where

acceptance is based on how close the multiplier is to zero). It is shown that the powers of the three tests are comparable.

The test is extended to the situation where the information matrix is singular. It is also extended to situations where several random variables are being observed, but not all with the same number of observations. The author gives one example, namely, a test for homogeneity in a 2×2 contingency table. It turns out that the Lagrange multiplier statistic is the usual one used in the χ^2-test for homogeneity. More informative examples are promised in a subsequent paper. *D. H. Shaffer* (Pittsburgh, Pa.)

3063:

Fabian, Václav. Some modifications of interval estimation and choice of number of observations for a binomial random variable. Apl. Mat. **4** (1959), 35–52. (Czech. Russian and English summaries)

Some modifications of interval estimation for binomial random variables are described, and tables of suitable sample sizes constructed.

V. E. Beneš (Murray Hill, N.J.)

3064:

Finney, D. J. The efficiencies of alternative estimators for an asymptotic regression equation. Biometrika **45** (1958), 370–388.

In estimating the parameters of equation $E\{y\}=\alpha-\beta\rho^x$, expressing the dependence of y on an independent variate x in terms of parameters α, β, and ρ, p presents the chief problem. A number of useful suggestions have been made by Stevens (1951), Pimentel Gomes (1953) and Patterson (1956) for the important special case of equal numbers of observations at each of several equally spaced values of x, with equal variance for all observations.

The primary purpose of this paper is to study the relations between the Patterson method and alternatives relative to interval regression termed by Hartley (1948). Two models (constant variance model and increasing variance model) of generation of error are also considered. The author confines his discussion to the case of four observations equally spaced in respect to x, and shows that under this particular situation, the Patterson estimator is always a fairly safe one to use, because its efficiency is never low and it is unlikely to be seriously biased. The content of the paper implies Patterson's linear estimator, Taylor's estimator, Hartley's estimators, and numerical comparisons in terms of the asymptotic variances of various estimators for the two variance models. *T. Kitagawa* (Fukuoka)

3065:

Tate, R. F. Unbiased estimation: Functions of location and scale parameters. Ann. Math. Statist. **30** (1959), 341–366.

Unbiased estimators for functions of a location parameter θ and a scale parameter ρ are expressed as unknown functions in integral equations of convolution type, and then obtained by integral transform methods including Laplace, bilateral Laplace and Mellin transform. The main results consist in the application of various derived expressions to the exponential distribution with parameters θ and ρ, the gamma and Weinbull distributions with parameter ρ, and to general distributions with truncation parameter θ. The methods used in this paper are related with those of Washio, Morimoto and Ikeda and hence with the operational calculus approach by Kitagawa.

The present paper confines itself to the functions of location parameter and/or of scale parameter, but treats cases in which the range of the density depends on the parameter.

In Section 4 a formula is derived which provides unbiased estimators for many functions $\zeta(\rho)$ with Mellin transforms, including the case $\xi(\rho)=\Pr\{X\in A|\rho\}$. Section 5 contains the results which provide minimum variance unbiased estimators for a wide class of functions $\xi(\theta)$. Section 6 is devoted to the case of a translation parameter. Very important special cases including exponential, gamma, and Weinbull distributions are discussed in Section 7, and are shown to have intimate connections with the works by Birnbaum and Sauders, Epstein and Sobel. *T. Kitagawa* (Fukuoka)

3066:

Dorogovcev, A. Ya. Confidence intervals in estimation of parameters. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 355–358. (Ukrainian. Russian and English summaries)

Asymptotic expansion of a confidence interval based on the maximum likelihood estimator.

J. Wolfowitz (Ithaca, N.Y.)

3067:

Bartholomew, D. J. A test of homogeneity for ordered alternatives. Biometrika **46** (1959), no. 1/2, 36–48.

Consider k normal distributions having means $m_1, m_2, \cdots, m_k$ and variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2$. In this paper the author tests the null hypothesis,

$$H_0: m_1 = m_2 = \cdots = m_k$$

against the "ordered alternative" hypothesis

$$H_2: m_1 \geqq m_2 \geqq \cdots \geqq m_k.$$

Tests of H_0 against H_2 are based on the likelihood ratio test. The tests are related to the familiar χ^2 and F criteria which are used when testing H_0 against the alternative

$$H_1: m_1 \neq m_2 \neq \cdots \neq m_k.$$

In testing H_0 against H_2 one is led to a modified χ^2 test, if the σ_i^2 are all assumed to be known, and to a modified F test if $\sigma_i^2=\sigma^2/n_i$, where the n_i are known sample sizes and where σ^2 are estimated from the data. For the case $k=2$, one tests H_0 against H_2 by using a one-tailed test; but for $k>2$, the distinction between one- and two-tailed tests is lost. One can consider this paper, therefore, as a generalization of the one-tail test.

It is difficult to find general solutions. The author succeeds in finding the distribution of a modified χ_k^2 statistic for $k=3$ and 4, and gives a partial result for $k=5$. In a note added in proof, he indicates that a recently established recurrence formula makes it possible to give the distribution of the modified χ_k^2, for $k>5$, in the case where all of the σ_i^2 equal a common σ^2. The problems involved in finding the distribution of the modified F_k test are considered briefly. Tables for the 5% and 1% points

of the modified χ_k^2 are given. Several well-chosen numerical examples are given to show how the modified χ_k^2 and F_k tests can be applied.

Benjamin Epstein (Stanford, Calif.)

3068:

Mickey, Ray. Some bounds on the distribution functions of the largest and smallest roots of normal determinantal equations. Ann. Math. Statist. **30** (1959), 242–243.

Let S_{ij}^1 and S_{ij}^2 $(i, j=1, 2, \cdots, k)$ be two sample covariance matrices from normal distributions having identical covariance matrices. In this note the author gives a lower bound on the distribution function of the smallest root, W_1, and an upper bound on the distribution function of the largest root, W_k, of the equation $|S_{ij}^1 - WS_{ij}^2| = 0$. *Benjamin Epstein* (Stanford, Calif.)

3069:

Wijsman, Robert A. Applications of a certain representation of the Wishart matrix. Ann. Math. Statist. **30** (1959), 597–601.

Let A be a Wishart matrix, i.e., the second order sample moment matrix (with respect to the origin) of an n-sample from a p-variate normal $(0, \Sigma)$ distribution. It is known that this matrix may be written (1) $A = CTT'C'$, where $CC' = \Sigma$ and T is a triangular stochastic matrix with independent elements; the diagonal elements of T are χ-distributed with varying degrees of freedom, whereas the off-diagonal elements are normal $(0, 1)$. The author shows that (1) leads to simple representations, in terms of independent variables, of several important multivariate statistics, such as the simple and multiple correlation coefficients and Anderson's sphericity criterion.

G. Elfving (Helsingfors)

3070:

Theiler, G. Une proposition de la théorie de la statistique non paramétrique. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 481–490. (Romanian. Russian and French summaries)

The author denotes by $S_m(x)$ and $T_n(x)$ the so-called empirical repartition functions of the independent random variables X and Y having the same theoretical repartition law $F(x)$, and proves that

$$\lim P[-\beta(m) < S_m(x) - T_n(x) < \alpha(m)] = \lim P[-\beta(m) < F(x) - T_m(x) < \alpha(m)]$$

when $m \to \infty$ and $\alpha(m) \to \alpha(>0)$, $\beta(m) \to \beta(>0)$. A corresponding proposition is enunciated for the exterior of the interval $(-\beta, \alpha)$ as well as for the intervals $(-\infty, -\beta)$, $(-\beta, +\infty)$, $(\alpha, +\infty)$, etc. These theorems are generalizations of a proposition assumed as being valid by B. C. Koroliuk.

O. Onicescu (Bucharest)

3071:

Konijn, H. S. Some nonparametric tests for treatment effects in paired replications. J. Indian Soc. Agric. Statist. **9** (1957), 145–167.

Consider pairs of random variables (W_i, V_i), $i = 1, 2, \cdots, n$. A survey of non-parametric tests of the hypothesis that the distribution of each W_i is the same as that of the corresponding V_i against translation alternatives is given along with a survey of non-parametric tests for the two-sample location problem. Under the assumption that $W_i, V_i - \theta_i/\sqrt{n}$ have the same distribution and of independence between pairs, the asymptotic distributions of the statistics of the randomized t, sign, Mann-Whitney, rank-sum, and median tests are derived and formulas for the corresponding asymptotic power functions for translation alternatives are given. Asymptotic efficiencies are computed in certain cases.

F. C. Andrews (Eugene, Ore.)

3072:

Federer, Walter T. Variance and covariance analyses for unbalanced classifications. Biometrics **13** (1957), 333–362.

The purpose of this paper is to present additional results beyond those obtained by various authors for variance and covariance analyses with unbalanced classifications, including Cases I, II and III. Variance analysis in Case I (Interaction absent) is known as the method of fitting constants, that in Case II (Interaction present; the effects assumed to be fixed effects) as the weighted squares of mean analysis, while Case III (Interaction present; the interaction effects and at least one of the main effects of the factors represented in the interactions assumed to be random effects) has been treated for balanced classifications but not for unbalanced ones. The additional results included in the present paper represent extensions of previous works by M. N. Das [J. Indian Soc. Agric. Statist. **5** (1953), 161–178; MR **16**, 942] and W. T. Federer [Cornell Univ. Bull. **BU-51-M** (1954) (mimeo); *Experimental design; theory and application*, MacMillan, New York, 1955] to unbalanced q-way classifications with several covariates. The statistical procedures for the three cases have been derived for two-way and three-way classifications and are illustrated with numerical examples for the two-way classification with a covariate. The procedures for a q-way classification with b covariates are also indicated.

T. Kitagawa (Fukuoka)

3073:

Dehalu, M. La formule de R. A. Fisher relative au coefficient de regression multiple. Note préliminaire. Bull. Soc. Roy. Sci. Liège **27** (1958), 171–184.

3074:

Baten, W. D.; and Frame, J. S. The polynomial correlation coefficient. Amer. Math. Monthly **66** (1959), 283–287.

It is assumed that the variates y and x are related in such a way that the regression of y on x is the polynomial $f(x) = \sum_{i=0}^{N} a_i x^i$. Let a set of n x's be given in their own standard units such that their first $2N$ moments are equal to those for a normally distributed variable and let the accompanying y's in the n pairs, (x^i, y^i), also be expressed in their own (sample) standard units. The square of the correlation coefficient determined from the sample is defined by $r^2 = 1 - \sum (f-y)2/n$, in which r^2 is maximized by fitting $f(x)$ to the data, i.e., determining the a_i, by least squares. It is then shown that $r^2 = \sum^N [\overline{f^{(i)}(x)}]^2/i!$, in which $\overline{f^{(i)}(x)}$ is the average of the values of the ith derivative of the fitted $f(x)$ at the given values of x. This result is illustrated with a numerical example with $N = 2$.

C. C. Craig (Ann Arbor, Mich.)

3075:

Cox, C. P. A concise derivation of general orthogonal polynomials. J. Roy. Statist. Soc. Ser. B. **20** (1958), 406–407.

The author gives the orthogonal polynomials corresponding to a general regression equation by means of a Schweinsian expansion of the determinantal form of the general regression equation. *K. Endl* (Columbus, Ohio)

3076:

Patterson, H. D. The use of autoregression in fitting an exponential curve. Biometrika **45** (1958), 389–400.

The behaviour of the simple regression $y_{x+1}=\alpha(1-\rho)+\rho y_x$ of y_{x+1} on y_x as an estimate of ρ in the equation $y_x=\alpha-\beta\rho^x$, where $0<\rho<1$, $x=0, 1, 2, \cdots, n-1$ and the y_x are subject to independent errors with equal variance, is discussed. In contrast to the result pointed out by Finney (1958) that the estimate is never of low efficiency and is unlikely to be subject to a serious bias when $n=4$, the author shows that, as n increases, the efficiency decreases markedly over the useful range of ρ and the estimate is subject to an increasingly large negative bias to the effect that the general use of this estimate can not be recommended. Consequently, the author discusses a more general regression equation

$$y_{x+1}=\frac{\alpha k(1-\rho)}{k+l\rho}+\frac{\rho}{k+l\rho}(ky_x+ly_{x+1}),$$

where $k\neq 0$. Alternative estimates given by the regression of y_{x+1} on ky_x+ly_{x+1} have been considered, and are shown to have the same efficiency as the simple regression, to have reduced biases by a suitable choice of k/l, and to be of limited value because of the difficulty of choice of k/l unless n is small. Numerical results are shown for these estimates with low bias. *T. Kitagawa* (Fukuoka)

3077:

Raghavarao, Damaraju. Some optimum weighing designs. Ann. Math. Statist. **30** (1959), 295–303.

N objects are to be weighed in N weighings on a chemical balance having no bias. Consider the non-singular design matrix $X(N\times N)=(x_{ij})$, where $x_{ij}=-1, +1$, or 0, if the jth object of the ith weighing is respectively on the right pan or the left pan or not weighed. The regression model for the recorded result, y_i, of the ith weighing is $\sum_j x_{ij}w_j=y_i+\varepsilon_i$. The method of least squares is used to obtain estimated weights, $\hat{w}_j$. If Var $(\varepsilon_i)=\sigma^2$, the variance-covariance matrix for the $\hat{w}_j$ is $(X'X)^{-1}\sigma^2=(c_{ij})\sigma^2$. Results in this paper are restricted to designs for which all c_{ii} are equal and all c_{ij} $(i\neq j)$ are equal.

Best weighing designs (defined as those for which $\sum c_{ii}$ is minimized), meeting the above conditions, are shown to be restricted to $X=P_N$ or S_N, where $P_N'P_N=(N-1)I_N+E_{NN}$ and $S_N'S_N=(N-1)I_N$; I_N is the identity matrix of order N and E_{NN} an Nth order matrix with positive unit elements everywhere. A necessary condition for the existence of P_N is that $2N=d^2+1$, where d is an odd integer. Certain balanced incomplete block designs can be used to construct P_N matrices. For S_N, $N=p^h+1$, where p is an odd prime and $p^h\equiv 1 \pmod 4$ for h a positive integer. Numerical examples are given for $N=5$, 6, 10, 13, 14, 18 and 25. *R. L. Anderson* (Raleigh, N.C.)

3078:

★Connor, W. S.; and Zelen, Marvin. Fractional factorial experiment designs for factors at three levels. National Bureau of Standards Applied Mathematics Series, No. 54. U.S. Government Printing Office, Washington, D.C., 1959. v+37 pp. $0.30.

This publication is a sequel to *Fractional factorial experiment designs for factors at two levels* [Nat. Bur. Standards Appl. Math. Ser. No. 48, 1957; MR **19**, 189], which contains plans for factors at two levels. This new document catalogues 41 fractional factorial experiment plans for experiments having 4 to 10 factors each at three levels, and for fractions ranging from 1/3 to 1/243 of the possible treatment combinations for experimentation. The treatment combinations are also arranged into blocks of 3, 9 or 27 experimental units each. These experiment designs have been constructed so as (1) to have no main effects "aliased" (or confounded) with block effects or with two-factor interactions and (2) to have as few as possible two-factor interaction effects "aliased" with other two-factor interaction effects. Each experiment design contains (i) the treatment combinations written out in full, (ii) the fundamental identity, and (iii) information as to those two-factor interaction effects which are "aliased" with higher-order interaction effects. A section on analysis discusses the estimation and biases of the various main effects and interactions. An example is presented of a 1/9 fractional replicate of a 6-factor experiment for a completely randomized and for a 27-block design. *R. L. Anderson* (Raleigh, N.C.)

3079:

Kiefer, J.; and Wolfowitz, J. Optimum designs in regression problems. Ann. Math. Statist. **30** (1959), 271–294.

The authors consider the problem of computing optimal designs for estimating coefficients in linear regression problems. Suppose that for any point x, one can observe a random variable Y_x for which $E(Y_x)=\sum a_i f_i(x)$ and Var $Y_x=\sigma^2$ where the f_i $(i=1, 2, \cdots, k)$ are known functions. The problem of selecting N x's so as to estimate one coefficient a_k optimally is first considered. Three approaches are discussed and related. These are the direct, the Chebyshev, and the game theoretic approaches. The main computational technique is that of reducing the problem to a Chebyshev approximation problem (for which the solution is often in the literature) and to the solution of $k-1$ simultaneous linear equations. The game theoretical approach is used mainly, but not exclusively, for theoretical reasons. The main result is based on the fact that for large N the solution may be approximated by one where all the observations are distributed among the x's which maximize

$$\left|f_k(x)-\sum_1^{k-1} c_j^* f_j(x)\right|,$$

where the c_j^* are chosen to minimize this maximum. (The c_j^* are the solution of the Chebyshev approximation problem.) Using game theoretic results, the authors present a generalization of Elfving's result that an optimum design may be selected using at most k x's. Further results are obtained for the case where it is desired to estimate several coefficients or the whole regression function. These results involve the criteria of

optimality. The reviewer is not in great sympathy with the authors' invariance arguments for the use of the *D*-optimality criterion. This is the more or less standard one of minimizing the determinant of the covariance matrix of the coefficients being estimated.

H. Chernoff (Stanford, Calif.)

3080:

Higuti, Isao. Note on the decision of optimal tolerance in the design of a simple random assembling. Ann. Inst. Statist. Math., Tokyo **8** (1956), 113–118.

3081:

★Chernoff, Herman; and Moses, Lincoln E. Elementary decision theory. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xv+364 pp. $7.50.

This is a book about decision making, under uncertainty about the state of nature, for a student with only a background of U.S. high school mathematics. The first six chapters deal with the processing of data (graphical methods, means and variances), probability and random variables, the concept of utility (treated axiomatically following von Neumann) and the comparison of various strategies (Bayes, minimax etc.). The remaining four chapters provide an introduction to classical statistics (the adjective is the authors') and discuss point and interval estimation and tests of hypotheses. There are numerous appendixes in which side-issues or more difficult points are discussed, and a good set of the usual tables which appear at the end of most statistical text books.

The first six chapters form an excellent introduction to decision theory. Great care has been taken to explain the basic ideas. The mathematics is as simple as the authors claim but symbols are very freely used. In marked contrast the examples and exercises are in personal terms ("Mr. Butregs owns a cow . . ."), a style that I found unpleasant. Decision theory should be much more widely taught as part of a statistics or operational research course than it is, and the present book provides an admirable basis for this.

The remaining four chapters are, because the first part has been so good, a disappointment. My impression is that the authors felt obliged to include standard statistical techniques (such as confidence intervals) because they are used in practice, but were out of sympathy with them because they do not fit into the decision framework (cf. their remarks on p. 215). The current outlook in statistics is not in decision terms, and it is doubtful whether it should ever be.

A puzzling feature of the book is that, although utility is treated axiomatically, prior probabilities are not. Why did they reject Savage's extension of the argument to degrees of belief? Had they not done so they could have confined attention to Bayes' strategies and could have worked out a satisfactory account of statistics which fitted nicely into the framework of the first part. Much of this has already been done by Jeffreys [*Theory of probability*, 2nd ed., Oxford University Press, 1948; for a review of the 1st, 1939, edition, see MR **1**, 151]. His results agree well with current practice.

D. V. Lindley (Cambridge, England)

3082:

Mikami, Misao. On a multi-level sampling inspection plan for continuous production. Bull. Math. Statist. **7** (1956), 1–10.

3083:

Wang, Shou-jen. On the estimation of regression coefficients of a random field with homogeneous residual. Acta Math. Sinica **8** (1958), 210–221. (Chinese. English summary)

"By our method we can extend the asymptotic efficiency formula to more general types of spectral functions of the residual field." (From the author's summary)

K. L. Chung (Syracuse, N.Y.)

3084:

Phillips, A. W. The estimation of parameters in systems of stochastic differential equations. Biometrika **46** (1959), no. 1/2, 67–76.

Let $\xi(t)$ and $\eta(t)$ be vector valued processes whose components are stationary and stationarily correlated. A linear differential relation with constant coefficients is assumed to exist between the two processes. The author derives expressions for the joint covariance functions and their Laplace transforms (or spectral densities), assuming that the given spectra are rational functions of frequency. Periodic sampling from a continuous time process is discussed. The main result of the paper is a discussion of how and when the equations can be identified; in the case when they are not all of the same order this is made possible by assuming that the disturbances are not cross-correlated.

U. Grenander (Stockholm)

3085:

Striebel, Charlotte T. Densities for stochastic processes. Ann. Math. Statist. **30** (1959), 559–567.

Consider a stochastic process $x(t)$ with two hypothetical probability measures P_1 and P_2. To test the hypothesis P_1 against P_2 it is useful to know when one measure is absolutely continuous with respect to the other and to calculate in a constructive way the corresponding Radon-Nikodym derivative. We meet the same problem in estimation. It was shown by the reviewer [Ark. Mat. **1** (1950), 195–277; MR **12**, 511] that this could be done in terms of the usual likelihood ratio. In the present paper it is shown how the problem can be studied with advantage from the point of view of martingale theory. A criterion for the existence of the derivative is given and the author gives a condition for when the density can simply be computed for a countable set of time points. These results are applied to certain Gaussian processes, among others the Orstein-Uhlenbeck process.

U. Grenander (Stockholm)

3086:

Svešnikov, A. A. Determination of the probability characteristics of three-dimensional sea-waves. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 32–41. (Russian)

The author discusses the relative merits of two methods of determining the spectral density of the random velocity potential characterizing three-dimensional wave motion of the sea; he assumes that the available data consists of

experimental records of the function $\zeta(x, y, t)$, the departure from its equilibrium position of the height of the sea's surface measured at the point (x, y) and the time t. One method uses one-dimensional auto- and cross-correlation of $\zeta(x, y, t)$ and the partial derivatives $\zeta_x(x, y, t)$ and $\zeta_y(x, y, t)$, taken for fixed (x, y); the other method uses two-dimensional spatial auto-correlation of $\zeta(x, y, t)$, taken for fixed t. *R. A. Silverman* (New York, N.Y.)

NUMERICAL METHODS

See also 2061, 2726, 2727, 2783, 3043, 3130, 3132, 3221, 3237, 3273.

3087:

Zuhovic'kiĭ, S. I.; and Leonova, V. B. Programming of an algorithm for Čebyšev approximation for a finite system of inconsistent linear equations. Luc'kiĭ Deržˇ. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **6** (1958), no. 3, 21–36. (Ukrainian)

The algorithm described here is given as part of an algorithm for determining the coefficients in a Chebychev sense of the approximation $\sum_{k=1}^{n} \xi_k \varphi_k(q)$ to a continuous function $f(q)$ over an interval Q, where the n functions $\varphi_k(q)$ are to be used as approximants. Thus the values ξ_k are to be found to yield

$$\inf_{\xi_k} \max_{q \in Q} \left| \sum_{k=1}^{n} \xi_k \varphi_k(q) - f(q) \right|.$$

This problem depends on a "best approximation" solution (given here in detail) to

$$\sum_{k=1}^{n} a_{ik} \xi_k = b_i \quad (i = 1, \cdots, m),$$

where $m > n$, yielding those values of ξ_k to give

$$\inf_{\xi_k} \max_{q \in Q} \left| \sum_{k=1}^{n} a_{ik} \xi_k - b_i \right|.$$

It is assumed the nth order minors of the determinant of the $m \times n$ matrix have non-zero determinant; and an algorithm for a series of approximations, based on a gradient method, is described in detail. The program for solution on a three-address instruction digital computer, the M-2, is described, with a flow diagram and portions of the coding included. An incomplete description of the M-2 is given. *J. W. Carr, III* (Chapel Hill, N.C.)

3088:

★Ziller, A. Calcul de la valeur numérique de la dérivée seconde, pour une valeur de la variable, d'une fonction donnée par des valeurs non équidistantes. Actes des colloques de calcul numérique, Caen, 1955; Dijon, 1956; pp. 61–85. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 77, Paris, 1958. vi + 144 pp. 2105 francs.

The starting point of the article is the expression for the second derivative of Newton's fundamental interpolation formulae truncated before the third-order term. The roots of the third-order equation arising from equating to zero the first neglected term are called "favourable values". By many examples it is shown that in case of the favourable values the errors are less than for other values.

The well-known nomogram for the solution of a third-order equation is included. *W. H. Muller* (The Hague)

3089:

★Sard, Arthur. The rationale of approximation. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 191–207. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U.S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x + 462 pp. (1 insert) $4.50.

Exposé général, suivant les vues actuelles, de la théorie de l'approximation des fonctionnelles (linéaires le plus souvent) des éléments contenus dans un ensemble donné d'un espace donné, ou de fonctionnelles aléatoires.

Dans le cas des espaces hilbertiens, l'auteur rappelle les notions d'efficience et d'efficience stricte qu'il a introduites et étudiées dans J. Math. Phys. **35** (1956), 127–144 [MR **18**, 322].

Il rappelle aussi les résultats qu'il a obtenus dans les espaces de fonctions continues ainsi que certaines de leurs dérivées partielles, sur un segment de R^n [Proc. Sympos. Appl. Math. Vol. VI, Numerical Analysis, pp. 177–185, Amer. Math. Soc., Providence, R.I., 1956; MR **18**, 33]. *J. Favard* (Paris)

3090:

★Egervary, B.; Lovass-Nagy, V.; and Kolin, L. Discrete models and matrix methods in engineering mechanics. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 259–276. Indian Society of Theoretical and Applied Mechanics. Indian Institute of Technology, Kharagpur, 1958. xi + 362 pp.

Methods of matrix inversion are developed to solve Poisson's and the biharmonic equation, replacing the continuum by a discrete network. The operations involved in the inversion of an $n \times m$-th order matrix are much reduced in number if the matrix can be partitioned into n^2 mth order blocks, commutable in pairs. Use is made of this in developing a calculation procedure suitable for programming. The method of reduction is extended from a rectangular to an arbitrary network, and some numerical examples are given. *J. Heyman* (Cambridge, England)

3091:

Farnell, A. B. Curve-fitting matrices. Amer. Math. Monthly **66** (1959), 297–300.

Two procedures are given for the inverse of a Vandermonde matrix of order n where the elements x_j need not be equidistant. Such matrices may be used in curve-fitting procedures.

The most suitable method involves simple recurrence relations and is thus eminently suitable for digital computers. The other method, suggested by the referee, consists in forming the matrix C whose elements are given by

$$C_{ij} = \begin{cases} p_n'(x_i)/(x_i - x_j)p_n'(x_j) & (i \neq j), \\ p_n''(x_j)/2p_n'(x_j) & (i = j), \end{cases}$$

where $p_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$. Repeated powers of the matrix C enable the inverse of the Vandermonde matrix to be obtained. *D. C. Gilles* (Glasgow)

3092:

Roth, J. Paul. An application of algebraic topology: Kron's method of tearing. Quart. Appl. Math. **17** (1959), 1–24.

In previous papers [Proc. Nat. Acad. Sci. U.S.A., **41** (1955), 518–521, 599–600; MR **17**, 536] the author considered the network problem and tearing in an abstract setting. After reviewing and completing his previous results he now considers in detail the more concrete situation that arises if suitable bases are chosen in the vector spaces involved and matrices are used to represent the linear mappings. The problem becomes that of solving a system of n equations

$$\begin{bmatrix} i \\ I \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} \begin{bmatrix} e \\ E \end{bmatrix}$$

(or the equivalent inverse system), where the components $i=(i_1, \cdots, i_p)$ and $E=(E_{p+1}, \cdots, E_n)$ are the unknowns. In this situation and devoid of geometry (which is not the author's approach), Kron's method of tearing for finding the necessary inverse of Y_4 is essentially to replace some of the known components of e by parameters and, after making a change in base, inverting a larger (but possibly more amenable) submatrix. Having done this, one returns to the original base and eliminates the parameters. In the last three sections, examples are considered in detail and tearing is compared with three more standard techniques for solving the network problem.

{There are two minor errors in the paper: 1. In equation (7.5), $Q\begin{bmatrix} i' \\ I \end{bmatrix} = I_\# \neq I^\# = I_\# - Y_3{}^\# e^\#$; 2. In equation (8.5) read "$-A^{-1}BR$" instead of "$-BR$" in the upper right of the matrix.}

J. B. Giever (University Park, N.M.)

3093:

★Fazekas, F.; und Sándor, I. Lösung der geodätischen Normalgleichungen nach einem neuen Iterationsverfahren. Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/57, pp. 37–46. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

The main purpose of this paper is to report on a method for solving a system of n linear equations, $Ax=b$, which was developed by J. Egerváry [Magyar Ind. Akad. Mat. Kutató Int. Közl. **1** (1956), 109–123]. After a review of the standard method of least square approximation and the resulting normal equations, the method of Egerváry is formulated and applied to a numerical example. The method is an iterative algorithm yielding the solution in n steps. At step k, an auxiliary $(n+1)\times(n+1)$ symmetric matrix Y_k is calculated from Y_{k-1} and from the kth row of the given augmented matrix (A, b). Starting with the identity $Y_0=I$, the rank of Y_k is reduced by one in each step, the nth step yielding a matrix Y_n which is essentially the dyadic product of the solution vector x with itself. No proofs or properties of this process with regard to error accumulation are given. It is remarked that the method of Hestenes-Stiefel is a special case of Egerváry's method. In the reviewer's opinion, there is no support for this statement. There appears to be no way of obtaining any information about the solution at an intermediate step of the method under discussion. However, the method is suitable for machine computation. The number of multiplications involved is of the same order as in the Hestenes-Stiefel method ($O(n^3)$), while storage space is less, since only one row vector of (A, b) need be available in the machine at each step and since the Y_k are symmetric.

H. J. Weinitschke (Los Angeles, Calif.)

3094:

★Fazekas, F.; und Sándor, I. Anwendung eines neuen rangvermindernden Verfahrens bei den Randwertaufgaben langer Kreiszylinderschalen. Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/57, pp. 55–73. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

The authors recommend using the "rank-diminishing" iterative method for solving a system of linear algebraic equations due to Egerváry [see previous review] in boundary value problems arising in shell theory. The method is formulated and illustrated by solving—by separation of variables—a well known 8th order partial differential equation for the bending of a cylindrical shell. Appropriate boundary conditions then lead to a system of linear equations for the determination of the coefficients in the complementary solution.

H. J. Weinitschke (Los Angeles, Calif.)

3095:

Appl, F. C.; and Zorowski, C. F. Upper and lower bounds for special eigenvalues. J. Appl. Méch. **26** (1959), 246–250.

A method is presented for finding upper and lower bounds for the fundamental eigenvalue in special eigenvalue problems. A power series is used as an approximation for the eigenfunction; the series is required to satisfy all the boundary conditions, but not the differential equation, and to make an error function and some of its derivatives vanish at each boundary. Using the enclosure theorem of L. Collatz [*Eigenwertaufgaben mit technischen Anwendungen*, Akad. Verlagsges., Leipzig, 1949; MR **11**, 137; p. 126], upper and lower bounds for the eigenvalue are generated, and it is shown that under certain conditions these bounds converge from above and below to the exact eigenvalue.

G. B. Warburton (Edinburgh)

3096:

Tarnove, Ivin. Determination of eigenvalues of matrices having polynomial elements. J. Soc. Indust. Appl. Math. **6** (1958), 163–171.

A description is given of a numerical method for finding roots λ of the equation $|A(\lambda)|=0$, where $A(\lambda)$ is given as an $n\times n$ matrix whose elements are polynomials in λ.

The author uses an iterative root-finding technique previously discussed by the reviewer [Math. Tables Aids Comput. **10** (1956), 208–215; MR **18**, 766]. Although this method requires but a single evaluation of the determinant $|A(\lambda)|$ per iteration, the results of the last three iterations are required during the computation. Therefore, to start the iterative procedure, it is necessary to select three initial approximations to a given root. The author proposes the selection of the points $\lambda^{(1)}=(1+\beta)u$, $\lambda^{(2)}=(1-\beta)u$, and $\lambda^{(3)}=u$, where u is chosen as the best initial estimate to the root, and $\beta<1$ is chosen to reflect the accuracy of this estimate. If an accurate initial estimate u is provided, this starting procedure is claimed by the

author to enhance the rate of convergence and to permit the evaluation of only the desired roots.

Applications of this method to the solution of control systems are discussed and questions of scaling and accuracy are mentioned although the details of the numerical method for evaluating the determinant $|A(\lambda)|$ are omitted from the discussion.

D. E. Muller (Urbana, Ill.)

3097:

Derr, John I. A unified process for the evaluation of the zeros of polynomials over the complex number field. Math. Tables Aids Comput. **13** (1959), 29–36.

The process is based on the recurrence relation

$$z_{i+1} = z_i - (k_i - l_i)f^{(l_i)}(z_i)/f^{(l_i+1)}(z_i),$$

with an explicit method for determining l_i and k_i. It is shown that if the sequence of numbers z_i, determined by this process, converges to a root γ of multiplicity k of $f(z)$, then, for all sufficiently large i, $k_i = k$ and $l_i = k-1$. Thus, at the later stages of the iteration, Newton-Raphson's method is being used to determine γ as a simple root of $f^{(k-1)}(z)$. This gives quadratic convergence and fair accuracy, whatever the multiplicity of γ. There is a tendency for closely bunched roots to be approximated by a single root of higher multiplicity. Some solutions obtained on the RAND corporation JOHNIAC computer are discussed to illustrate the process.

A. M. Duguid (Providence, R.I.)

3098:

Householder, Alston S. Dandelin, Lobačevskiĭ, or Graeffe? Amer. Math. Monthly **66** (1959), 464–466.

Discussion of historical propriety of the term "Graeffe's method".

3099:

Greenspan, Donald. On popular methods and extant problems in the solution of polynomial equations. Math. Mag. **31** (1957/58), 239–253.

The author describes a variety of numerical and a few graphical methods. The bibliography contains sixty-one references. *S. Gorn* (Philadelphia, Pa.)

3100:

Schmidtmayer, Josef. Efficient solution of algebraic equations of 3rd through 5th degree (with real coefficients). Pokroky Mat. Fys. Astr. **3** (1958), 659–671. (Czech)

In this expository article the author tries to demonstrate by numerical examples that the most primitive formal and graphical devices can, if properly applied, lead to good approximations of roots of equations of 3rd, 4th, and 5th degree. [There are references to Cardano (1545), Vieta (1615), Descartes (1637), Todhunter (1867), Matthiessen (1878): *Grundzüge der antiken und modernen Algebra der litteralen Gleichungen*, Petersen (1897), and some modern Czech writings on the classical theory of equations.]

H. Schwerdtfeger (Montreal, P.Q.)

3101:

Tricomi, Francesco G. Sul resto delle formule di quadratura numerica migliorate col metodo di "estrapolazione". Boll. Un. Mat. Ital. (3) **14** (1959), 102–104. (English summary)

In this short paper the expression and an upper bound for the remainder of certain "improved" numerical quadrature formulas are given, and especially for the "improved" Simpson's formula. *Author's summary*

3102:

Gautschi, Walter. Exponential integral $\int_1^\infty e^{-xt}t^{-n}dt$ for large values of n. J. Res. Nat. Bur. Standards **62** (1959), 123–125.

An asymptotic expansion for the integral in the title is derived, and numerical bounds for the error are given. The terms of the series are rational functions of n and x, and the remainder after the kth term is $O(n^{-k})$.

W. Wasow (Madison, Wis.)

3103:

Miller, James; and Hurst, R. P. Simplified calculation of the exponential integral. Math. Tables Aids Comput. **12** (1958), 187–193.

Values of the exponential integral may be obtained over the entire range of argument by means of a tabulated value and an interpolation formula based on Taylor's expansion of the slowly varying exponential product function $-e^x E_i(-x)$. The accompanying table gives $\pm E_i(\pm x)$, $\pm e^{\pm x}E_i(\pm x)$ for

$$x = .2(.05)5(.1)10(.2)20(.5)50(1)80$$

to 16D. *I. A. Stegun* (Washington, D.C.)

3104:

Kis, Ottó. Notes about convergence of the interpolatory method for the approximate solution of differential and integral equations. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 25–41. (Russian. Hungarian and English summaries)

The author considers an interpolation method for approximate solution of differential and integral equations. The integral equation (1) $\varphi(x) = \lambda \int_a^b K(x,t)\varphi(t)dt + f(x)$ (λ not an eigenvalue) is solved approximately by a polynomial $\psi_n(x) = \sum_{k=1}^n c_k x^{k-1}$, the coefficients being determined by the requirement that (1) be satisfied at a set of n points, $x_1, x_2, \cdots, x_n$ of the interval $[a, b]$. It is shown that for sufficiently large n, ψ_n exists uniquely. If $R(F)$ denotes the difference between a function $F(x)$ and its Lagrange interpolation polynomial based on $x_1, \cdots, x_n$, it is shown that $\max_{a\le x\le b}|\psi_n - \varphi| = O(\max|R(K)| + \max|R(f)|)$ and that $\|\psi_n - \varphi\|_{L^2} = O(\|R(K)\|_{L^2} + \|R(f)\|_{L^2})$. More accurate estimates can be given if the x_k are chosen to be the Čebyšev or Gaussian points. Using a result of Erdös and Turan [Ann. of Math. (2) **38** (1937), 142–155] it is shown that if for $\psi_n(x)$, the x_k are chosen to be the zeros of $\omega_n(x)$, the L^2-mean error is $O(E_n(K) + E_n(f))$. Here $\{\omega_n(x)\}$ is the set of polynomials orthogonal on $[a, b]$ with respect to a weight function $p(x) \ge m > 0$, generated from $1, x, x^2, \cdots$. $E_n(F)$ is the error of the best possible approximation to F by a polynomial of degree n.

The author also considers linear differential equations

$$y^{(2m)}(x) = \sum_{k=0}^{2m-2} f_k(x)y^{(k)}(x) + f(x),$$

(2)

$$y^{(k)}(a) = y^{(k)}(b) = 0, \qquad k = 0, 1, \cdots, 2m-1.$$

The homogeneous system is assumed to have no nontrivial solution. Here the solution y is approximated by $z_n(x)=(b-x)^m(x-a)^m\sum_{k=1}^{n}c_kx^{k-1}$, the coefficients c_k being determined by requiring (2) to be satisfied at the points $x_1, x_2, \cdots, x_n$. It is shown that if the x_k are the roots of $\omega_n(x)$ as above, then z_n exists uniquely for sufficiently large n, and that $z_n^{(k)}$ converges uniformly to $y^{(k)}$ for $k=0, 1, \cdots, 2m-1$, $z_n^{(2m)}$ converging in the mean to $y^{(2m)}$. Estimates for the error are given. In particular if the coefficients $f, f_0, f_1, \cdots, f_{2m-2}$ have r derivatives which satisfy Lipschitz conditions of order α, then the uniform error of $z_n^{(k)}$ and the L_2-mean error of $z^{(2m)}$ are all $O(n^{-r-\alpha})$. *W. S. Loud* (Minneapolis, Minn.)

3105:

Franklin, J. N. Numerical stability in digital and analog computation for diffusion problems. J. Math. Phys. **37** (1959), 305–315.

The author considers a class of problems involving the diffusion equation $u_t=Au$, where the differential operation A has the form $A=\nabla\cdot(\alpha(x)\nabla)-\beta(x)$, and where $\alpha(x)$ and $\beta(x)$ are positive and non-negative scalar functions, respectively, of the one-, two-, or three-dimensional vector x. Solutions are desired which have prescribed values for $t=t_0$ in a bounded region of space V and which satisfy given linear auxiliary conditions on the boundary of V, for all $t\geq t_0$. The methods considered involve the replacement of the differential operator A by a difference operator A_m defined on a suitable set of mesh points of V. The resulting difference-differential equation $v_t=A_mv$ can be solved by analogue methods by considering the related system of ordinary differential equations $dw/dt=B_mw$, where w is a vector with one component for each mesh point of V and where B_m is a matrix related to the difference operator A_m. The author shows that in analogue computation stability is assured. The proof would have perhaps been clearer if the author had stated the exact mathematical implications of various electrical terms such as "passive networks" and "ground capacitors."

For digital computations, the time derivative, as well as the space differential operator, is replaced by a difference quotient. The use of the forward difference quotient leads to a method frequently known as the "forward difference method." One can calculate

$$v^{n+1} = v(x, t+(n+1)\Delta t)$$

explicitly in terms of v^n. However, for stability it is ordinarily necessary for the condition $\Delta t=O(\|\Delta x\|^2)$ to be satisfied, a requirement which frequently involves a large amount of computational effort. With implicit schemes such as the backward difference method and the Crank-Nicolson method there is no restriction, as far as stability and convergence are concerned, on the manner in which Δt and $\|\Delta x\|$ go to zero, according to Lax and Richtmyer [Comm. Pure Appl. Math. **9** (1956), 267–293; MR **18**, 48]. However, although the implicit procedures are easy to carry out in one dimension, they are more difficult to apply in higher dimensions. The author considers explicit operators of the form $v^{n+1}=p_k(\Delta tB_n)v^n$ where p_k is a polynomial of degree k so chosen as to maximize the value of Δt which may be used subject to the requirement that the method be stable. The best polynomial of degree k is given by $E_k(z)=(-1)^k2^{2k-1}T_{2k}(k^{-1}(\frac{1}{2}z)^{1/2})$ where, for each integer n, $T_n(x)$ is the Tschebyscheff polynomial $T_n(z)=2^{1-n}\cos(n \arccos x)$. For the diffusion equation $u_t=u_{xx}$, the explicit method so obtained is stable provided $\Delta t\leq(k^2/2)(\Delta x)^2$. This is a considerable improvement over the requirement for the forward difference method where the condition $\Delta t\leq(\frac{1}{2})(\Delta x)^2$ must hold. The author states that the theoretical improvement has been observed in actual numerical cases. On the other hand, Δt must still decrease like $(\Delta x)^2$. Therefore, it would seem that implicit methods might be preferable even for two- or three-dimensional problems, particularly since there are rapidly convergent iterative methods available for solving the implicit equations. Other methods which seem more promising include the alternating direction method of Peaceman and Rachford [J. Soc. Indust. Appl. Math. **3** (1945), 28–41; MR **17**, 196] and Douglas [ibid. 42–65; MR **17**, 196] for two-dimensional problems and a similar method of Douglas and Rachford [Trans. Amer. Math. Soc. **82** (1956), 421–439; MR **18**, 827] for two- and three-dimensional problems. *D. M. Young, Jr.* (Austin, Tex.)

3106:

Saul'ev, V. K. On estimation of the error in finding eigenfunctions by the method of finite differences. Vyčisl. Mat. **1** (1957), 87–115. (Russian)

The elliptic partial differential characteristic value problem

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+\lambda u = 0 \quad \text{in } Q,$$

$$\left(\sum_{i,j=1}^{n}a_{ij}\zeta_i\zeta_j \geq \nu\sum_{i=1}^{n}\zeta_i^2,\ \nu>0\right), \quad u|_\Gamma = 0,$$

is approximated by using first order finite (forward and backward) differences to approximate the partial derivatives. A theorem is given, in the n-dimensional case, for upper and lower bounds on the pth characteristic value in the approximation as a function of p and n, using for comparison the corresponding characteristic values for the original problem and that given by the Laplace differential operator over inscribed and circumscribed cubical regions.

A second theorem, again for the n-dimensional case, bounds the maximum absolute values of the pth normalized characteristic function of the finite difference approximant in terms of p, n, and $1/h$. For a special case with $n=2$, a special argument sharpens the latter bounds. The final result, for $n=2$, shows the maximum value of the absolute difference between the true and approximating pth normalized characteristic function to be of $O(h^2)$ and gives the corresponding multiplier as a function of the coefficients a_{ij}, the geometry of the region, the derivatives of the first eight orders of the true solution, the distance of the true pth characteristic value from other true characteristic values, and the approximating pth characteristic value.

J. W. Carr, III (Chapel Hill, N.C.)

3107:

Yuan', Čžao-Din. Convergence of certain difference schemes for the heat conduction equation. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 2, 9–14. (Russian)

The author considers two unstable difference methods for the approximate solution of the heat equation in

$0<x<1$, $t>0$ subject to $u(0,t)=u(1,t)=0$ and $u(x,0)=f(x)$ for $0\leq x\leq 1$. Both methods have been studied by W. Leutert [Proc. Amer. Math. Soc. **2** (1951), 433–439; J. Math. Phys. **30** (1952), 245–251; MR **13**, 287, 783] who showed that by appropriately choosing the initial function, corresponding to f, in the difference problem one can obtain convergence as $\Delta t=r(\Delta x)^2\to 0$, where r is any positive constant and Δt, Δx are the mesh widths. The author extends these results to the case $\Delta t=r\Delta x\to 0$. He also establishes similar convergence results for difference-differential approximations such as

$$(\Delta t)^{-1}\{u(x,t+\Delta t)-u(x,t)\} = \partial^2 u/\partial x^2$$

as $\Delta t\to 0$. Similar approximation methods for the initial value problem are also treated.

D. G. Aronson (Minneapolis, Minn.)

3108:

Bellman, R.; Cherry, I.; and Wing, G. M. A note on the numerical integration of a class of non-linear hyperbolic equations. Quart. Appl. Math. **16** (1958), 181–183.

The object of this paper is to present a method for the numerical integration of certain non-linear hyperbolic equations (resembling hydrodynamic equations), which will result in a good representation of the solution over large regions, and will not be adversely effected by the presence of "shock" discontinuities. The method is illustrated by the numerical integration of

$$(1) \qquad u_t = -uu_x, \quad u(x,0)=f(x)$$

(for specific $f(x)$), using the finite difference equation

$$(2) \quad u(x,t+\Delta)=u(x-u(x,t)\Delta,t), \quad u(x,0)=f(x),$$

where t assumes values $0, \Delta, 2\Delta, \cdots$, and x assumes values $\cdots, -2\delta, -\delta, 0, \delta, 2\delta, \cdots$. No special relation between δ and Δ is assumed. When the values $u(n\delta,t)$ have been found for $t=\tilde{t}$, the values for $t=\tilde{t}+\Delta$ are found by using the equation

$$u(n\delta-u(n\delta,t)\Delta,\tilde{t}) = u(n\delta,\tilde{t}+\Delta).$$

(Expanding equation (2) in powers of Δ shows that it is equivalent to (1) to first-order terms in Δ.)

The results of various numerical integrations of this form are compared with values of an analytic solution, revealing excellent agreement.

Extensions of this method to general systems of hyperbolic equations in one space variable have not succeeded in avoiding instability. *R. B. Davis* (Syracuse, N.Y.)

3109:

★**Marchuk, Gurii Ivanovich. Numerical methods for nuclear reactor calculations.** Translated from the Russian: Supplement nos. 3–4 of the Soviet Journal of Atomic Energy, Atomnaya Energiya, Atomic Press, Moscow, 1958. Consultants Bureau, Inc., New York; Chapman & Hall, Ltd., London; 1959. vi+295 pp. Paperbound: $60.00.

This important book fills a real gap in the literature pertaining to the theory of neutron chain reactors and nuclear reactors. Among others, Glasstone and Edlund [*The elements of nuclear reactor theory*, Van Nostrand, New York, 1952] furnish a comprehensive survey of the elementary aspects of the theory; Davison and Sykes [*Neutron transport theory*, Clarendon Press, Oxford, 1957; MR **20** #2217] a mathematically and physically sophisticated view of the problem of neutron transport; and Weinberg and Wigner [*The physical theory of neutron chain reactors*, Univ. of Chicago Press, Chicago, Ill., 1958] a scholarly treatment of the physics of neutron chain reactors. None of these books attempts to discuss the problems arising from the actual calculation of the performance of a chain-reactor system. The book under review does just this, and very successfully.

The author begins with the derivation of the Boltzmann equation for neutron transport. Then he derives the various approximations (diffusion, age-diffusion, etc.) that have been found useful for the simplification of this equation. In each such case, the difference equations which approximate the differential (or integral) equation of the problem are derived and carefully discussed, and methods developed for their solution. A striking feature of the work is the deserved emphasis on the set of equations adjoint to the original set; these adjoint equations are treated, in every case, in parallel with the originals, and every method of solution of the original equations is accompanied by a method for the solution of the adjoint equations.

As the author himself emphasizes, this book is not the place to acquire an understanding of the basic principles of nuclear reactor theory, a prerequisite for its use is a study of one or all of the books mentioned above.

If the book has a real weakness, it lies in the physics; not all the evaluations of the approximations involved would be agreed to by other workers in this field. But, given a system of equations from which to work, one will find here many fruitful methods for translating these equations into concrete results.

The author states correctly that the book should be valuable to graduate students, engineers, and specialists in the field of nuclear reactor calculation; it is unfortunate that the unavoidably high price of the translation will severely limit its use. *R. R. Coveyou* (Oak Ridge, Tenn.)

COMPUTING MACHINES

See also 3087, 3287.

3110:

Sauer, Robert. Sulla calcolatrice elettronica PERM (Politecnico di Monaco:Bav.) e sullo sviluppo della matematica numerica. Rend. Sem. Mat. Fis. Milano **28** (1959), 3–17. (English summary)

A brief discussion of the fundamentals of computers and of programming, illustrated by reference to the Munich Polytechnic Institute's PERM computer. This is a parallel machine, having a drum memory of 8192 50-bit words, with 2 millisecond access time. Its features include indirect addressing and B-register control.

W. J. Turanski (Philadelphia, Pa.)

3111:

Alterman, Z.; and Rabinowitz, P. An interpretive system on the WEIZAC. Bull. Res. Council Israel. Sect. F **7** (1957/58), 136–138.

MECHANICS OF PARTICLES AND SYSTEMS

3112:

Gulmanelli, P.; e Loinger, A. Formulazione gruppale della dinamica. Rend. Sem. Mat. Fis. Milano **28** (1959), 87–106. (English summary)

Recent physical theories, particularly quantum mechanics, have placed great emphasis on the group-theoretical structure of the equations of mathematical physics. This approach originated in classical dynamics, but was formulated more broadly as a mathematical theory by F. Klein and by S. Lie. The groups of greatest interest at present in physical theory are the Lorentz group and its subgroups in 4-dimensional space-time.

The authors review this theory in some detail, particularly for the case of the classical mechanics of particles. They base their interpretation on the circumstance that translations and rotations of the space-time reference system can be regarded as canonical transformations. The construction of the generating functions of the transformations yields the energy, momentum, and angular momentum variables. Indications of the extension of the theory to relativistic mechanics and to field theories are given. {This line of argument has been developed in greater detail for field theories by J. M. Jauch [An. Acad. Brasil. Ci. **20** (1948), 353–361; MR **10**, 498].}

E. L. Hill (Minneapolis, Minn.)

3113:

Kirgetov, V. I. On transpositional relations in mechanics. J. Appl. Math. Mech. **22** (1958), 682–693 (490–498 Prikl. Mat. Meh.).

For a system with k generalized coordinates subjected to constraints expressed by $m<k$ equations linear in the velocities $\dot{q}$, the paper gives the following theorems.

Every kinematically admissible motion passing through a given point can be represented by a one-parameter family

$$q_\nu = \phi_\nu(t, \alpha), \quad \frac{\partial^2\phi_\nu}{\partial\alpha\partial t} \equiv \frac{\partial^2\phi_\nu}{\partial t\partial\alpha},$$

where $k-m$ of the functions ϕ_ν can be arbitrarily chosen.

If the variation of every motion included in all possible families of kinematically admissible motions satisfies all possible transpositional relations, then the equations of constraints are completely integrable.

The author establishes conditions under which the time derivative of the chain of possible displacements at a certain point is determined, and the necessary and sufficient conditions in order to have

$$\delta\dot{q}_{s+i} = \frac{d}{dt}\,\delta q_{s+i} \quad (i = 1, 2, \cdots, m-s).$$

E. B. Schieldrop (Oslo)

3114:

Nožička, František. Les lignes géodésiques, asymptotiques et les lignes de courbure du point de vue de la mécanique d'un point matériel. Apl. Mat. **4** (1959), 83–108. (Czech. Russian and French summaries)

Eine Fläche $x^\alpha = x^\alpha(\eta^a)$ $(\alpha = 1, 2, 3;\ a = 1, 2)$ sei in einem Kraftfeld F^α eingebettet. Das Kraftfeld wirkt auf einen materiellen Punkt der Masse m, dessen Bewegungen auf der Fläche verlaufen müssen. Dann gelten für die Bewegungen dieses Punktes unter dem Einfluß des Kraftfeldes die Differentialgleichungen

$$m\nabla_t \frac{d\eta^a}{dt} = B_\alpha{}^a F^\alpha \quad (a = 1, 2)$$

($\nabla_t d\eta^a/dt$ Abkürzung für

$$\frac{d^2\eta^a}{dt^2} + \left\{ \begin{matrix} a \\ cb \end{matrix} \right\} \frac{d\eta^a}{dt}\frac{d\eta^b}{dt},$$

t Zeit, $B_\alpha{}^a = \bar{g}^{ac} g_{\alpha\beta} B_c{}^\beta$, $B_\alpha{}^a = \partial x^\alpha/\partial\eta^a$) und die Bestimmung der Bewegung wird eindeutig mit den Anfangsbedingungen

$$(\eta^a)_{t=t_0} = \eta_0{}^a, \quad \left(\frac{d\eta^a}{dt}\right)_{t=t_0} = v_0{}^a.$$

Man kann nun die differentialgeometrisch ausgezeichneten Kurven der Fläche vom Standpunkt dieser Punktmechanik untersuchen. In Abwesenheit äußerer Kräfte bleibt der materielle Punkt auf der Fläche entweder in Ruhe oder in Bewegung längs einer geodätischen Linie. Ist die Bahnkurve eine Asymptotenlinie der Fläche, so verschwindet der Druck des Punktes auf die Fläche. Erweist sich dagegen dieser Druck (bei passender Definition) längs der Bahnkurve als Extremalwert, so ist die Bahnkurve eine Krümmungslinie der Fläche.

M. Pinl (Cologne)

3115:

Myard, Francis. Engrenages hélicoïdaux externes de mêmes sens et vitesse angulaire. C. R. Acad. Sci. Paris **248** (1959), 3404–3405.

L'auteur montre qu'on peut créer (en dépit d'une invraisemblance a priori) des engrenages externes tournant dans le même sens.

Résumé de l'auteur

3116:

Mangeron, D.; Drăgan, C.; und Svijevschi, Vl. Eine neue Methode zum Studium der Theorie der Mechanismen und Maschinen. Rev. Méc. Appl. **2** (1957), no. 1, 145–156.

In den Somowschen Formeln für die höheren Beschleunigungen eines Punktes in der ebenen Kinematik begegnet man den Grössen A_i und B_i definiert durch $A_{i+1} = \dot{A}_i + \dot{\theta}B_i$, $B_{i+1} = \dot{B}_i - \dot{\theta}A_i$, $A_1 = \dot{\theta}^2$, $B_1 = \ddot{\theta}$. Der erste Verfasser hat den Begriff der reduzierten Beschleunigung höherer Ordnung eingeführt, nämlich die Beschleunigung i-ter Ordnung geteilt durch A_i. Es gilt der Satz: der geometrische Ort der Endpunkte reduzierter Beschleunigungen der Punkte einer Geraden l ist eine Gerade, welche auf l senkrecht steht. Mittels dieses Satzes wird für $i = 1, 2$ die Beschleunigungsverteilung der Ebene behandelt und eine Reihe von Aufgaben aus der Getriebelehre konstruktiv gelöst.

O. Bottema (Delft)

3117:

Bugaievski, E.; Bogdan, R.; und Pelecudi, C. Zur Einteilung räumlicher Mechanismen. Rev. Méc. Appl. **2** (1957), no. 1, 157–170.

Ausgangspunkt dieser Betrachtungen ist die Formel von Dobrowolski {auch wohl nach Grübler genannt, Rev.} für den Beweglichkeitsgrad einer kinematischen Kette als Funktion der Gliederzahl und der Klassenzahlen der Elementenpaare. Der Ersatz eines Gelenkes durch ein

anderes wird betrachtet, die Anzahl der geschlossenen Polygone, der Einfluss der Beseitigung von Gliedern und Gelenken, u.s.w. und dann eine Theorie der Aequivalenz für Getriebe aufgebaut, welche zu einer Einteilung führt, die in einer Tafel anschaulich gemacht wird.

O. Bottema (Delft)

3118:

Meyer zur Capellen, W. Die zweidimensionale Fourieranalyse spezieller Koppelkurven. Z. Angew. Math. Mech. **39** (1959), 31–40. (English, French and Russian summaries)

The harmonic analysis of the equations of motion of the kinematic linkages, carried out in a series of papers by the author, has given important results, particularly from the dynamical point of view. The analysis is now extended to the components of the coupling rod curves; for example, to determine the locus of the center of mass. Such an analysis consists basically in establishing the Fourier series for the cosine and sine of the coupling rod angle. The coefficients of both series are determined by the type and dimensions of the linkage and are mutually correlated. Certain particular cases, which give very simple solutions, are discussed.

E. Leimanis (Vancouver, B.C.)

3119:

Onicescu, Octav. Sur la mécanique du point matériel. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 461–465.

The author here shows that his new mechanics [Rev. Math. Pures Appl. **2** (1957), 167–180; MR **20** #2223] leads to the usual results when applied to a free particle and to a particle in an electromagnetic field.

A. J. Coleman (Toronto, Ont.)

3120:

Valcovici, Victor. Sur les liaisons holonomes et non holonomes. Rev. Méc. Appl. **3** (1958), no. 4, 365–371.

L'auteur a donné ailleurs [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **102** (1958), no. 4; C. R. Acad. Sci. Paris **243** (1956), 1012–1014; MR **19**, 784; **18**, 428] une théorie sur les liaisons dans la mécanique des systèmes qui part du fait que c'est dans la nature des choses de ne pas donner une relation pour les vitesses mais pour les déplacements. Ici il généralise ses idées, sortant d'une analyse du processus du mouvement en mécanique et il montre qu'une liaison doit se réduire à une forme lineaire de dq et dt, dont les coefficients sont des fonctions de t, q, $\dot{q}$, et $\ddot{q}$.

O. Bottema (Delft)

3121:

Onicescu, O. Die Mechanik des starren Körpers. Rev. Méc. Appl. **3** (1958), no. 3, 179–184.

Es werden die allgemeinen Gleichungen der Bewegung eines starren Körpers abgeleitet, der sich in einem Feld befindet und gewissen Bindungen unterliegt.

W. Zerna (Hanover)

3122:

Kusnetsoff, L. I. Estimation of the solutions of the equations of motion of gyroscopic systems. Vestnik Leningrad. Univ. **14** (1959), no. 7, 105–111. (Russian. English summary)

The author considers small oscillations of a dynamical system involving m gyroscopes. The kinetic energy, in generalized coordinates, is assumed to depend quadratically on the position and velocity coordinates. Using the angular momentum integrals one obtains a system of second order linear differential equations (S) for the position coordinates. In the "simplified equations" the positional accelerations are neglected. The author estimates the deviation of the solutions of the "simplified equations" from the corresponding solutions of (S).

L. Markus (Minneapolis, Minn.)

3123:

Coppel, W. A. On the equation of a synchronous motor. Quart. J. Mech. Appl. Math. **12** (1959), 242–256.

The author studies the real differential equation

$$\ddot{x}+\alpha\dot{x}+\sin x = \beta, \tag{1}$$

where α and β are positive constants. He writes $\ddot{x}+\sin x = \alpha(\eta-\dot{x})$ and solves $\ddot{x}+\sin x=0$ by elliptic integrals. For small $\alpha>0$, the author uses the approximations of slowly varying phase and amplitude to study the asymptotic behavior of the solutions. The author summarizes his conclusions:

If $\eta<4/\pi$, all solutions converge to a finite limit as $t\to+\infty$, but if $\eta>4/\pi$, only those solutions converge for which initially $\dot{x}<0$ or $\sin^2\frac{1}{2}x+\frac{1}{4}\dot{x}^2\leq 1$. The remaining solutions behave asymptotically as though x were the angle made with the downward vertical by a simple pendulum of length g which performs complete revolutions in a time $2k_0K(k_0)$, where k_0 is the unique root between 0 and 1 of the equation $E(k)=\frac{1}{4}\pi\eta k$.

L. Markus (Minneapolis, Minn.)

3124:

Rumiantsev, V. V. [Rumyancev, V. V.] On the stability of motion of a gyroscope of gimbals. II. J. Appl. Math. Mech. **22** (1958), 694–700 (499–503 Prikl. Mat. Meh.).

[For part I, see same J. **22** (1958), 513–520; MR **21** #1719.]

The author investigates the stability of a certain kind of motion of a symmetrical gyroscope on gimbals when the stationary axis of the outer gimbal ring is horizontal.

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 3040, 3105, 3109, 3278.

3125:

Galkin, V. S. On a class of solutions of Grad's moment equations. J. Appl. Math. Mech. **22** (1958), 532–536 (386–389 Prikl. Mat. Meh.).

The author notes an interesting generalization of the exact solutions of the exact equations of moments obtained independently by himself [Prikl. Mat. Meh. **20** (1956), 445–446; MR **19**, 1102] and, in somewhat greater generality, by the reviewer [J. Rational Mech. Anal. **5** (1956), 55–128; MR **17**, 796; see Equations (32.1)]. In these flows the velocity field is spatially linear, with time-dependent coefficients, while moments of all orders other than 1 are functions of time only. For these flows, Grad's 13-moment approximation yields the exact values of the stresses, but generally not of the higher moments. The author notes some simple cases {but does not work out

enough of the physical or mathematical properties of the solutions to give much idea of what phenomena occur}.

C. Truesdell (Bloomington, Ind.)

3126:

Friedman, Harold L. On Mayer's ionic solution theory. Molecular Phys. **2** (1959), 23–38.

This paper extends the McMillan-Mayer theory of electrolytes [J. Chem. Phys. **13** (1945), 276–305] by including 3, 4, ···-body terms in the potential of average force between a set of ions in the solvent. The resulting expansion for the grand canonical partition function is analyzed in terms of properly defined graphs.

L. Van Hove (Utrecht)

3127:

Silin, V. P. The oscillations of a degenerate electron fluid. Soviet Physics. JETP **35** (8) (1959), 870–874 (1243–1250 Ž. Eksper. Teoret. Fiz.).

The collective oscillations of a degenerate electron fluid are treated by Landau's semi-classical theory of Fermi fluids. The Boltzmann equation is written down for the distribution function $n_\sigma(r, p)$ of the particles in phase space and spin, and linearized. Effects taken into account are the coupling with macroscopic electric and magnetic fields, the adiabatic interactions of the particles by Landau's phenomenological method, and the collision operator. The Boltzmann equation is rewritten in terms of the phase space density $f \equiv Sp_{\sigma'} n_{\sigma'}(r, p)$, and the spin density $\sigma \equiv Sp\hat{\sigma}n$. Specific problems discussed are plane wave oscillation of f and σ; general dispersion formulae are given and their consequences considered in the presence and absence of applied magnetic fields and with a variety of assumptions about the phenomenological interparticle interaction.

A. Herzenberg (Manchester)

3128:

Tiablikov, S. V.; and Tolmachev, V. V. Distribution functions for the classic electron gas. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1210–1213. (Russian)

3129:

Klimontovich, Iu. L.; and Temko, S. V. Kinetic equations for a system of particles interacting with lattice vibrations. Soviet Physics. JETP **35** (8) (1959), 799–803 (1141–1147 Ž. Eksper. Teoret. Fiz.).

A derivation is given of the kinetic equations for the distribution functions of electrons and vibrations of a crystalline lattice. In the spatially homogeneous case, the equation for the distribution function of the electrons is identical to the kinetic equation in Bloch's theory of electric conductivity. An equation is obtained for the lattice vibrations, and expressions are found for the frequency and the damping decrement of these vibrations.

Werner Nowacki (Bern)

3130:

Kofink, W. Studies of the spherical harmonics method in neutron transport theory. Nuovo Cimento (10) **9** (1958), supplemento, 497–541.

This paper deals with some approximate methods of solving the integrodifferential equations of the neutron transport theory for the plane case.

Three methods, all already known [Davison and Sykes, *Neutron transport theory*, Clarendon Press, Oxford, 1957; MR **20** #2217] are described and compared: (I) The spherical harmonics method; (II) The discrete ordinate method, with Gaussian quadrature formula; (III) Ditto with "interated angular distribution". (Here the spherical harmonics (s.h.) moments of the neutron angular distribution, i.e., flux, current, etc., are determined as in (II) and then substituted into the original equation. An ordinary differential equation results and this latter is solved exactly. This amounts to subtracting from (II) a certain correction term, which eliminates unwanted singularities. The procedure is due to Chandrasekhar [*Radiative transfer*, Oxford Univ. Press, 1950; MR **13**, 136].)

The answer obtained for the s.h. moments, i.e., for flux, current etc., by the three methods are identical, but not the answers given for the angular distribution as such. These latter agree only for a certain special set of directions.

With increasing order of approximation the expressions for flux, current, etc., converge [A. M. Weinberg and E. P. Wigner, *Physical theory of neutron chain reactors*, Univ. of Chicago Press, Chicago, Ill., 1958; H. B. Keller, J. Soc. Indust. Appl. Math. (to appear)], but in so far as the angular distribution as such (as a function of directional cosines) is concerned, the method (II) does not converge (except when it coincides with (III)). The author shows that (I) does not converge either, though in this case the lack of convergence is restricted to a narrow cone around the directions normal to the interface. On the other hand the method (III) is shown to converge.

B. Davison (Toronto, Ont.)

3131:

Birkhoff, Garrett. Reactor criticality in transport theory. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 567–569.

The study of branching processes, particularly those that arise in neutron transport theory and cosmic ray cascade theory, leads by way of functional equations for generating functions to the asymptotic behavior of various types of linear functional equations [cf. Bellman and Harris, same Proc. **34** (1948), 601–604; MR **10**, 311]. This asymptotic behavior is conveniently studied by means of the theory of positive operators, extensions of the matrix results of Perron and Frobenius and the integral kernel result of Jentzsch [cf. Bellman, *A survey of the mathematical theory of time-lag, retarded control and hereditary processes*, Rand Corporation, Santa Monica, Calif., 1954; MR **15**, 962; chapter 5]. Extensions of the Perron-Jentzsch results have been given by Krein and Rutman and by the author [Trans. Amer. Math. Soc. **85** (1957), 219–227; MR **19**, 296].

In this paper, a linear multiplicative (branching) process is defined in quite general terms and the results of the above-cited paper are used to derive some general theorems concerning asymptotic behavior. Further results of this nature will be found in a forthcoming paper by the author and in a forthcoming monograph by Harris on branching processes. *R. Bellman* (Santa Monica, Calif.)

3132:

Smets, Henri B. Un critère de stabilité non-linéaire des réacteurs nucléaires. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 102–107.

The author considers the stability of the solutions of a nonlinear functional equation which occurs in the theory of nuclear reactors, under a number of special assumptions concerning the form of the equation and the type of perturbation that is allowed.

R. Bellman (Santa Monica, Calif.)

3133:

Sack, R. A. Pressure-dependent partition functions. Molecular Phys. **2** (1959), 8–22.

The author gives a general discussion of the transition from statistical ensembles defined for given values of extensive parameters (energy, volume, etc.) to the ensembles corresponding to given values of the conjugate intensive parameters (temperature, pressure, etc.). He applies the general analysis to the problem of defining an ensemble and a partition function at constant pressure. Previous work on this subject gave rise to much confusion which is avoided by the unified treatment of the present paper. Special attention is devoted to ensembles defined entirely in terms of intensive variables and it is shown how to circumvent the difficulty that the extension of the system is then undetermined. The paper ends with illustrations of the general theory for perfect and imperfect gases.

L. Van Hove (Utrecht)

3134:

Ford, Joseph. Approach of one-dimensional systems to equilibrium. Phys. Rev. (2) **112** (1958), 1445–1451.

The author considers the (fine-grained) probability density in phase space for a finite system. It obeys the Liouville equation, which is integrated approximately for two simple one-dimensional systems (anharmonic oscillator and chain of anharmonic oscillators). Coarse-graining the approximate probability density thus obtained and assuming smooth initial conditions, the author shows that in the cases considered it approaches the equilibrium value in the limit of large times. It is suggested that a similar situation would hold for realistic systems.

L. Van Hove (Utrecht)

3135:

Brout, R.; and Prigogine, I. Statistical mechanics of irreversible processes. VIII. General theory of weakly coupled systems. Physica **22** (1956), 621–636.

This paper is part VIII of a series [see also Prigogine, Proc. Internat. Conf. Theoret. Phys., Kyoto-Tokyo, 1953, pp. 464–470, Science Council of Japan, Tokyo, 1954; Canad. J. Phys. **34** (1956), 1236–1245; MR **16**, 658; **18**, 611; Prigogine and Bingen, Physica **21** (1955), 299–311; MR **17**, 221] on irreversible processes in which a general theory based on classical mechanics is developed, corresponding to Van Hove's earlier quantal treatment [Physica **21** (1955), 517–540; MR **17**, 115]. The authors consider the evolution of a system with Hamiltonian $H=H_0+\lambda V$ consisting of N multiply-periodic subsystems with total Hamiltonian H_0 caused by their weak interaction λV; this evolution is described by Liouville's equation for the distribution function $\partial\rho/\partial t=[H,\rho]$. They claim that when $N\to\infty$ and $\lambda\to 0$ the system shows irreversible behaviour as $t\to\infty$ ($\lambda^2 t$ finite) so that, in particular, its entropy, defined in the usual way as the integral of $-k\rho\log\rho$ over all phase space, increases.

These conclusions contradict the generally accepted belief that neither classical nor quantum statistical mechanics can describe the irreversible evolution of a system without deliberately violating mechanics at some point by introducing an averaging procedure. The authors imply that the need for averaging can be replaced by the condition $N\to\infty$, and appeal to the work of Poincaré [*Méthodes nouvelles de la Mécanique Céleste* vol. 1, Gauthier-Villars, Paris, 1892]. Ingraham [see review below] has analyzed very carefully the theories of Van Hove and of Brout and Prigogine and finds that both are at fault, and for essentially the same reasons. It is quite possible that the equations of these theories have some physical significance, but it appears that they must be based on other premises.

W. Byers Brown (Manchester)

3136:

Ingraham, R. On the derivation of rate equations. Nuovo Cimento (10) **9** (1958), 99–124.

This is an important critical study of the theory of irreversible behaviour developed from quantum mechanics by Van Hove [Physica **21** (1955), 517–540; MR **17**, 115] and extended to classical mechanics by Brout and Prigogine [see preceding review]. Van Hove's theory leads directly to linear stochastic equations for the evolution of a slightly perturbed system with hamiltonian $H=H_0+\lambda V$ from a limiting form $\hat{U}(t)$ of the unitary transition operator $U(t)=\exp(-iHt/\hbar)$, without recourse to the usual averaging processes. The basic assumption about V, which Van Hove asserts is characteristic of large systems, is that the matrix elements $\langle E'\alpha'|VA|E\alpha\rangle$, where A is diagonal, have a diagonal singularity of the Dirac δ type. Ingraham's criticisms are: (a) that the treatment of the singular energy integrals which arise is physically incorrect, and violates the unitarity of $\hat{U}(t)$; (b) that the basic assumption does not characterise a large system correctly, and leads to an essential ambiguity in the improper integrals for $\hat{U}(t)$. The characteristics of a large system are reformulated and, by a more careful passage to the continuum limit, it is shown that $U(t)$ remains unitary and that stochastic behaviour does not occur.

An interesting feature of the paper, which may be useful to readers unfamiliar with the perturbation techniques for the transition operator $U(t)$ developed by quantum field theoreticians, is the lucid treatment of the simple example of a linearly perturbed oscillator by this method.

W. Byers Brown (Manchester)

3137:

Van Hove, Léon. The ergodic behaviour of quantum many-body systems. Physica **25** (1959), 268–276.

A generalisation of earlier work [Physica **21** (1955), 517–540; **23** (1957), 441–480; MR **17**, 115; **19**, 696] to show that quantities which are not diagonal in the unperturbed representation approach their equilibrium values under the action of a perturbation. The resulting ergodic theorem is discussed. The criticism of Ingraham [see review above] of the author's earlier papers is shown to be based upon an incorrect basic equation.

D. ter Haar (Oxford)

ELASTICITY, PLASTICITY

See also 3090, 3094, 3169, 3171.

3138:

Drucker, D. C. A definition of stable inelastic material. J. Appl. Mech. **26** (1959), 101–106.

A general definition for "stable" inelastic materials is proposed which is a reformulation in somewhat narrow non-thermodynamical terms of the well-known [H. von Helmholtz, *Wissenschaftliche Abhandlungen*, vol. 1, Barth, Leipzig, 1882; p. 223] principle of stationary rate of entropy production or energy dissipation, on the basis of which positive definite dissipation potentials can be postulated the existence of which ensures uniqueness of solutions of boundary-value problems in inelastic media [A. M. Freudenthal and H. Geiringer, *Handbuch der Physik*, vol. 6, pp. 229–433, Springer, Berlin, 1958; MR **20** #483; see pp. 251–256]. The concepts of convexity, normality and incremental linearity are consequences of this physical principle.

A. M. Freudenthal (New York, N.Y.)

3139:

Blokh, V. I. On the representation of the general solution of the basic equations of the static theory of elasticity for an isotropic body with the aid of harmonic functions. J. Appl. Math. Mech. **22** (1958), 659–668 (473–479 Prikl. Mat. Meh.).

3140:

Tolokonnikov, L. A. On the connection between stresses and strains in the non-linear theory of elasticity. Prikl. Mat. Meh. **20** (1956), 439–444. (Russian)

3141:

Gross, William A. The second fundamental problem of elasticity applied to a plane circular ring. Z. Angew. Math. Phys. **8** (1957), 71–73.

3142:

Miyao, Kazyu. Stresses in a circular disk with an eccentric circular hole fixed at its center and under a tangential force on the outer edge. Bull. JSME **2** (1959), 271–276.

3143:

Morgenstern, Dietrich. Mathematische Begründung der Scheibentheorie (zweidimensionale Elastizitätstheorie). Arch. Rational Mech. Anal. **3** (1959), 91–96.

Aus der dreidimensionalen Elastizitätstheorie gewinnt man durch anschauliche Betrachtungen die zweidimensionalen Gleichungen der Scheibentheorie. In der vorliegenden Untersuchung wird nachgewiesen, daß die Lösungen der dreidimensionalen Theorie für verschwindende Dicke der Scheibe gegen die Lösung der zweidimensionalen Theorie konvergieren.

W. Zerna (Hanover)

3144:

Toupin, R. A. Piezoelectric relations and the radial deformation of a polarized spherical shell. J. Acoust. Soc. Amer. **31** (1959), 315–318.

The author gives an elegant explicit solution to a boundary value problem of the linear theory of piezoelectricity, in which the stress tensor and the electric field vector are linear functions of the infinitesimal strain tensor and the electric displacement vector. He determines the displacement of a spherical shell, polarized in the radial direction and subject to prescribed uniform pressures and voltages at the inner and outer surfaces.

W. Noll (Pittsburgh, Pa.)

3145:

Ling, Chih-Bing. Stresses in a stretched slab having a spherical cavity. J. Appl. Mech. **26** (1959), 235–240.

Continuing his previous work [Quart. Appl. Math. **10** (1952), 149–156; **13** (1956), 381–391; MR **13**, 886; **19**, 902; Ling and Kuo-Liang Yang, J. Appl. Mech. **18** (1951), 367–370; MR **14**, 428] on spherical cavities in elastic bodies the author solves the problem of a cavity in a stretched slab under an all-round tension. A suitable disturbed biharmonic function and a biharmonic integral are added to the stress function for a slab without cavity. The disturbed function has a harmonic part of the type $r^n P_n$, $r^{-(n+1)} P_n$ and a biharmonic part of the type $r^{n+2} P_n$, $r^{-(n-1)} P_n$; the integral is formed in terms of $J_0(kr)$. The application of the Hankel transform helps to satisfy the boundary conditions. Two numerical examples are discussed in detail and it is shown that the maximum stress increases with the radius of the cavity.

B. R. Seth (Kharagpur)

3146:

★**Matheson, J. A. L. Hyperstatic structures: An introduction to the theory of statically indeterminate structures. Vol. I.** With chapters by N. W. Murray and R. K. Livesley. Academic Press, Inc., New York; Butterworths Scientific Publications, London; 1959. xv+474 pp. (6 plates) $15.50.

The structural engineer is concerned usually with the design criteria of strength and stiffness; the structure must carry the given loads (with some safety margin) and must meet certain deflexion requirements. An important third class of problem deals with stability, either of the structure as a whole or of its individual members.

In making an elastic analysis of the strength of a given structure, two distinct processes are involved, although these sometimes tend to get confused; the forces and/or bending moments at each section of the structure must first be determined, and then the resulting stresses calculated. The calculation of forces and bending moments is the process which qualifies for inclusion as a topic of theory of structures; stress calculation is more properly the concern of strength of materials. Now in the analysis of a statically determinate structure, calculation of forces and bending moments can be made solely by using equations of equilibrium; the analysis may be tedious, but is basically trivial. Theory of structures proper starts when statically indeterminate (or hyperstatic) structures are analysed, and equations of equilibrium alone no longer suffice for the solution. The book reviewed here deals with this topic of "advanced" structural analysis.

Professor Matheson starts by establishing clearly the two concepts of equilibrium and of compatibility of displacement, on which the whole theory of structures is based, and these two concepts are kept before the reader at every opportunity. The general theorems are developed

in this way, and it is pleasing to note how the principle of virtual work is used, and the importance assigned to the theorems of complementary energy. Professor Matheson is not concerned with expounding the trick methods of structural analysis, and, while full prominence is given to standard procedures, such as Castigliano's theorems, moment distribution, slope-deflexion equations, and so on, the reader is never put in the position of turning the handle in the hope of churning out an answer. Every procedure is closely and firmly related to the fundamental concepts.

It would be tedious to list the topics covered by Professor Matheson in this very comprehensive book; suffice it to say that the structural engineer will find almost anything he needs. There are omissions; for example, conjugate beam theory is not treated, but, on the other hand, the much more important topic of model analysis is given more space than it usually receives in structural textbooks. For this reviewer, the section devoted to plastic theory is disappointingly short.

Two former colleagues of Professor Matheson have contributed chapters. Dr. Murray has written a concise account of the stability of struts and frameworks, which draws on recent work in this rapidly developing field. Significantly, the book ends with a chapter on matrix methods by Dr. Livesley. These methods have not usually been stocked in the engineer's armoury, but the weapons are becoming more and more essential with the growing use of electronic computers.

Both in the choice of subject matter and in the way of presentation, Professor Matheson has managed to communicate in this book his great understanding of the behaviour of structures.

J. Heyman (Cambridge, England)

3147:

Atteia, Marc. Intégration des équations d'équilibre d'un cylindre de révolution encastré en ses deux extrémités et soumis à une pression hydrostatique. C. R. Acad. Sci. Paris **248** (1959), 3687–3689.

L'auteur montre que les équations d'équilibre d'un cylindre de révolution à axe horizontal encastré à ses deux extrémités, et soumis à une pression hydrostatique, s'intègrent exactement. *Résumé de l'auteur*

3148:

Romiti, Ario. Sul comportamento critico degli alberi rotanti aventi massa ed elasticità distribuite. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 174–183.

The author presents an excellent account and review of the theory of critical speeds of a rotating shaft in which various effects of nonuniform flexural rigidity, gravity, etc. are taken into account.

C. R. De Prima (Pasadena, Calif.)

3149:

Korbukova, L. D. Bending of a square anisotropic plate clamped at the edge. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 184–189. (Russian)

3150:

Tamate, Osamu; and Shioya, Shunsuke. On the transverse flexure of an infinite strip with semicircular notches on both edges. Bull. JSME **2** (1959), 264–271.

In this paper, the effects of a pair of symmetrical semicircular notches on the transverse flexure of an infinite strip under the state of plain bending are investigated by use of the Poisson-Kirchhoff theory of thin plates. The unknown coefficients included in the solution are determined by the method of perturbation.

From the authors' summary

3151:

Nash, W. A.; and Cooley, I. D. Large deflections of a clamped elliptical plate subjected to uniform pressure. J. Appl. Mech. **26** (1959), 291–293.

The von Karman large deflection equations are solved by a perturbation method based upon the smallness of the central deflection of the elliptical plate. The computational difficulties are not serious and involve considerably less labour than either of the two alternative approaches to this problem that are to be found in the literature [C. L. Perry, Proc. Sympos. Appl. Math., Vol. III, pp. 131–139, McGraw-Hill, New York, 1950; N. A. Weil and N. M. Newmark, J. Appl. Mech. **23** (1956), 21–26]. Results of the analysis are shown to be in excellent agreement with experimental data obtained from a test on an aluminium plate. *L. S. D. Morley* (Farnborough)

3152:

Au, T. Equations for thin toroidal shells. J. Aero/Space Sci. **26** (1959), 391–392.

Es wird gezeigt, daß sich Ringschalen mit Hilfe der Love-Meissnerschen Theorie der Rotationsschalen in ähnlicher Weise leicht lösen lassen wie Kegel- und Kugelschalen. Es werden die Differentialgleichungen der Ringschalen abgeleitet. Lösungen werden nicht angegeben. Für numerische Berechnungen wird die Differenzen-Methode empfohlen. *W. Zerna* (Hanover)

3153:

★Lakshmikantham, V.; and Ramakanth, J. Non-linear deformation of aeolotropic bodies. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 163–174. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi + 362 pp.

The finite deformation of a transversely isotropic spherical shell and a hollow circular cylinder subject to uniform normal surface tractions can be reduced to the solution of non-linear differential equations. The solution of such equations have been obtained graphically by W. M. Shepherd and B. R. Seth [Proc. Roy. Soc. London Ser. A **156** (1936), 171–192]. An alternative method is to obtain an analytic solution in the form of a power series, but, in general, it is not very easy to prove its convergence. The authors have obtained such a solution and by an ingeneous method due to one of the authors, V. Laksmikantham, have shown that the series is absolutely convergent.

For numerical results the graphical method seems to be more convenient. *B. R. Seth* (Kharagpur)

3154:

Stokey, W. F.; and Zorowski, C. F. Normal vibrations of a uniform plate carrying any number of finite masses. J. Appl. Mech. **26** (1959), 210–216.

This paper deals with the vibrations of a uniform plate carrying n added masses. The equations of motion are deduced from Lagrange's equations with the effect of the added masses included in the kinetic, but not in the potential, energy.

Expanding the displacement in a series of eigenfunctions of the unloaded plate and substituting into the equations of motion leads eventually to a determinantal equation of order n whose roots are the first n frequencies. The coefficients in this equation involve doubly infinite series, so in practice approximate frequencies are found by suitably truncating the series. As an example, the authors consider a simply supported rectangular plate with added masses of various magnitudes at the center and quarter span stations. Use of four terms in the above mentioned expansion to determine the fundamental frequency gives agreement with experiment generally better than 5%. In those cases where larger discrepancies occur, the authors ascribe the difficulties primarily to experimental defects. *W. E. Boyce* (Troy, N.Y.)

3155:

Vodička, Václav. A class of problems on longitudinal vibrations. Quart. Appl. Math. **16** (1958), 11–19.

This paper deals with longitudinal vibrations of composite bars, that is, bars composed of n parts with possibly different material properties. Laplace transform methods are employed, and the author obtains an expression for the transform of the axial displacement in the general case. The inverse transform is inaccessible in general, but has been carried out in three special cases (with n arbitrary): (i) vibrations of a bar under its own weight; (ii) vibrations of a bar with a mass on the end; (iii) vibrations due to collision of two bars. In all cases the solutions reduce to known results when $n=1$.

W. E. Boyce (Troy, N.Y.)

3156:

Institute of Mathematics, Academia Sinica. Mechanical vibrations and the problem of eigenvalues. Acta Math. Sinica **9** (1959), 17–27. (Chinese)

The usual classical equation of lateral vibration of a beam is solved for beams composed of two or more sections, each having a different flexural rigidity and different material density, by means of the method of determinants, the Rayleigh-Ritz-Galerkin method, and the method of finite differences.

Yi-Yuan Yu (Brooklyn, N.Y.)

3157:

Gusein-Zade, M. I. On the acoustic theory of spalling. J. Appl. Math. Mech. **22** (1958), 763–767 (547–549 Prikl. Mat. Meh.).

Considering the case of axial symmetry and unit step force applied to the front surface of a plate, a solution to the acoustic wave equation for displacement potential in the plate is developed by operational calculus. Contour integration yields stresses due to incident and reflected waves along the axis. Rear-face spalling occurs on wave reflection. A relation is developed for the depth of rear spalling for an arbitrary variation of surface force. A simplified relation is presented for the special case of a linearly-decaying saw-tooth pulse.

W. W. Soroka (Berkeley, Calif.)

3158:

Budiansky, Bernard. Extension of Michell's theorem to problems of plasticity and creep. Quart. Appl. Math. **16** (1958), 307–309.

The author proves that if the resultant force on each boundary of a multiply connected sheet of plastic material vanishes, then the solution of the plane stress problem is independent of Poisson's ratio. Possible resulting simplification in setting Poisson's ratio equal to $\frac{1}{2}$ is noted; also application to photoplasticity.

C. E. Pearson (Cambridge, Mass.)

3159:

Marciniak, Zdzisław. An analysis of the stability of a thin-walled shell subjected to tension in the plastic state. Rozprawy Inż. **6** (1958), 527–535. (Polish. Russian and English summaries)

The instability by lateral contraction or expansion of a thin-walled cylindrical tube of work-hardening material subject to an axial tensile force and internal pressure is analyzed on the basis of the Mises yield condition with graphically defined increasing yield limit. The plastic strain associated with instability is determined from the condition that the rate of increase of the second invariant of the stress deviation equals or exceeds the rate of increase of the square of the momentary yield limit.

A. M. Freudenthal (New York, N.Y.)

3160:

Ziegler, Hans. A modification of Prager's hardening rule. Quart. Appl. Math. **17** (1959), 55–65.

Prager has suggested a rule which translates the yield surface in stress space when plastic deformation takes place. A modification of this rule is proposed which translates the yield surface in the direction connecting its center with the stress point instead of in the direction of the strain increment. The vector from the origin to the stress point no longer represents the strain but more realistic hardening laws are possible for simple tension or compression. Also, any form of the proposed rule valid in a nine dimensional stress space is valid as well in six space or in any subspace. In particular, the deformation of the yield surface which occurs with the combination of Prager's rule and the Tresca criterion is avoided.

D. C. Drucker (Providence, R.I.)

3161:

Perrone, Nicholas; and Hodge, P. G., Jr. Strain-hardening solutions to plate problems. J. Appl. Mech. **26** (1959), 276–284.

Annular plates of rigid/plastic material are considered under uniform bending moments around the inner or outer edges. Stress profiles and deflected shapes are calculated with the Tresca yield condition and the Prager kinematic hardening rule. The computations are lengthy and it is shown that a fair approximation can be obtained, at least to the stresses, and with much less labour, by means of a modified hardening rule. This is anisotropic in character, but the inconsistency with postulated material isotropy in other respects is accepted as a fair price to pay.

R. Hill (Nottingham)

3162:
Ivlev, D. D. Discontinuous solutions of space problems in the theory of ideal plasticity. J. Appl. Math. Mech. **22** (1958), 669–677 (479–486 Prikl. Mat. Meh.).

Discontinuous fields of plastic stress are discussed from the purely statical point of view under the assumption that, at each point of the field, two of the principal shearing stresses have the critical value. It is not investigated to what extent these discontinuous stress fields are compatible with fields of plastic flow.

W. Prager (Providence, R.I.)

3163:
Scipio, L. Albert, II. On the design of structural models to study the thermal stress phenomenon. Ann. New York Acad. Sci. **79** (1959), 143–156.

3164:
Florence, A. L.; and Goodier, J. N. Thermal stress at spherical cavities and circular holes in uniform heat flow. J. Appl. Mech. **26** (1959), 293–294.

3165:
Biot, M. A. Further developments of new methods in heat-flow analysis. J. Aero/Space Sci. **26** (1959), 367–381.

Lagrangian methods in heat-flow problems and transport phenomena were introduced by the writer in some previous work [see J. Aero. Sci. **24** (1957), 857–873; MR **19**, 791]. The present paper develops further one particular aspect of the method, i.e., the elimination of "ignorable coordinates". This is accomplished by a special choice of generalized coordinates, each of which is constituted by an arbitrary temperature distribution and an "associated flow field". The latter is a vector field which is derived from the corresponding scalar field by a variational method. The procedure is valid for a certain class of non-linear problems, provided we replace the temperature by the heat content as the unknown. It is shown that for normal coordinates derivation of the associated flow field is immediate. The use of normal coordinates and their associated flow fields is illustrated by an example. Introduction of Dirac functions and associated flow fields yields a procedure which constitutes a generalization of the classical formulation by Green's functions and integral equations. This is illustrated by application to one-dimensional problems of heating of a homogeneous or composite slab and directly verified by classical methods in the Appendix. (From author's summary)

E. H. Mansfield (Farnborough)

3166:
Zeuli, Tino. Sforzi in una sfera omogenea isotropa dovuti a variazioni periodiche di temperatura sulla sua superficie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 366–382.

A homogeneous isotropic sphere is subjected to given periodic variations of temperature at its surface. The problem is to determine the temperature at an internal point and the internal stresses due to thermal dilatation. If there are no body forces, the problem is then to find the temperature T and the displacement $\mathbf{s}$ which satisfy the equations

$$\Delta_2 T - \frac{1}{k}\frac{\partial T}{\partial t} = 0,$$

$$\mu\Delta_2 \mathbf{s} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{s} - \frac{E}{1-2\sigma}\alpha_0 \operatorname{grad} T = 0,$$

given T on the surface of the sphere. Here α_0 is the linear coefficient of thermal expansion, E is Young's modulus, σ is Poisson's ratio and

$$\lambda = \frac{\sigma E}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)}$$

in the usual notation. *E. T. Copson* (St. Andrews)

FLUID MECHANICS, ACOUSTICS

See also 2830, 2892, 3086, 3157.

3167:
Moreau, Jean-Jacques. Une spécification du potentiel-vecteur en hydrodynamique. C. R. Acad. Sci. Paris **248** (1959), 3406–3408.

On montre que le champ de vitesses d'un fluide incompressible en présence de parois bornées fixes (sur un domaine borné ou non, peut-être multiplement connexe) possède un potentiel-vecteur à divergence nulle normal sur ces parois. Une fonction de Green est construite pour son calcul à partir de la vorticité; il en résulte une expression de l'énergie cinétique du fluide en fonction de la vorticité.

Résumé de l'auteur

3168:
Jones, E. E. The elliptic cylinder in a shear flow with hyperbolic velocity profile. Quart. J. Mech. Appl. Math. **12** (1959), 191–210.

For the steady plane flow of an inviscid liquid the vorticity is solely a function of the stream-function ψ. The essential assumption here is that far from any disturbing bodies the streamlines are parallel and that ψ is a linear combination of $\sinh Y'$ and $\cosh Y'$ where Y' is a non-dimensional coordinate perpendicular to the stream direction. The vorticity is then everywhere proportional to ψ and $\nabla_1^2\psi = k^2\psi$, a linear equation, so that the solution for an immersed body may be found by the superposition of a disturbance stream-function onto an explicitly known "undisturbed" stream-function.

For a body of elliptic cross-section, the use of elliptic coordinates results in a series solution in terms of Mathieu functions. For uniqueness it is assumed that the circulation around the curve coinciding with the profile in the disturbed stream has the same value as that around the same curve in the undisturbed stream. It is found that the stagnation streamline is displaced towards a region of higher velocity, the displacement increasing with incidence and non-uniformity of the main stream and decreasing with eccentricity. In addition the pressure, force and moment coefficients are obtained and investigated numerically for a flat plate, though the solution for the latter has infinite speed at both leading and trailing edges.

The reviewer believes that the circulation assumption is questionable. It is ascribed by the author to Tsien [Quart. Appl. Math. **1** (1943), 130–148; MR **5**, 21], although the latter merely implies it by default, and without discussion, in an introductory example. Physically it would appear to be unsound, since even with an irrotational main-stream a circulation develops due to asymmetry in initial shedding of vorticity. Also there seems to be no reason to suppose that the solution for even moderate angles of incidence will have any more physical meaning than in the irrotational case where the inviscid solution is invalidated by a large separated region of flow.

H. C. Levey (Nedlands)

3169:

Dean, W. R. An application in hydrodynamics of the Green's function of an elastic plate. Mathematika **5** (1958), 85–92.

The displacement of an infinite plate, bounded internally by a clamped circular edge, is obtained under given transverse forms. The solutions have a certain relevance to the theory of viscous flow past a circular cylinder which the author discusses briefly. *K. Stewartson* (Durham)

3170:

Toomre, Alar. A pulsating sphere in a rotating fluid. J. Fluid Mech. **5** (1959), 571–576.

"A pulsating sphere, which performs a sequence of virtually impulsive changes in its radius with time, is completely surrounded by an inviscid, incompressible fluid whose velocity field is generally rotational. This paper indicates how it is possible, by means of Helmholtz's theorem, to relate the corresponding vorticity and velocity fields immediately before and after such expansions or contractions.

The method is then applied to the case of a spherical mass of fluid initially in uniform rotation in which a spherical core undergoes a single sudden expansion, followed after a short interval by an equally rapid contraction back to the original radius. An interesting meridional flow is thereby induced, which tends to decrease the angular velocity of rotation of the fluid near the poles at the outer surface, relative to that of the equatorial fluid. It is perhaps significant that this is in qualitative agreement with the variation of angular velocities observed at the surface of the sun." (From the author's summary) *K. Stewartson* (Durham)

3171:

Khaskind, M. D. [Haskind, M. D.] Vibrations of thin plane wings in tandem in plane incompressible flow. J. Appl. Math. Mech. **22** (1958), 649–658 (465–472 Prikl. Mat. Meh.).

The solution of the title problem is obtained by treating separately the following two problems. The first is non-homogeneous and represents non-circulatory flow past a system of wings and the second is a homogeneous problem. This general case requires the solution of difficult integral equations in order to calculate some unknown constants. In order to proceed further, the author considers the special case of a tandem biplane in which one of the wings is fixed. For this special case, approximate expressions are derived for the hydrodynamic forces acting on the vibrating wing. *G. N. Lance* (Southampton)

3172:

Mangler, K. W.; and Smith, J. H. B. A theory of the flow past a slender delta wing with leading edge separation. Proc. Roy. Soc. London Ser. A **251** (1959), 200–217.

The problem stated in the title has been studied by a number of authors, including Roy [C. R. Acad. Sci Paris **234** (1952), 2501–2504; Recherche Aéronautique no. 56 (1957), 3–12], Legendre [ibid. no. 30 (1952), 3–8; no. 31 (1953), 3–6; no. 35 (1953), 3–6], C. E. Brown and W. H. Michael [J. Aero. Sci. **21** (1954), 690–694], Adams [ibid., **20** (1953), 430] and R. H. Edwards [ibid., **21** (1954), 134–135].

In the present paper a more elaborate model of the rolled-up, spiral vortex sheets above the wing is employed than in earlier investigations; namely, the inner part of the spiral, assumed small, is treated approximately, while the outer part is represented as a sheet whose shape and strength depends on a finite number of parameters. Numerical values of vortex-sheet shape and core position have been calculated, as well as the corresponding wing pressure distributions and normal force. Comparison of core positions and normal force with those of Brown and Michael show substantial differences, and the present results are seen to be closer to experimentally determined values. *W. R. Sears* (Ithaca, N.Y.)

3173:

Hacker, Tiberiu. Sur la stabilité longitudinale de l'avion lors des perturbations répétées. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 69–76. (Romanian. Russian and French summaries)

Pour traiter le problème du comportement de l'avion dans une atmosphère agitée (rafales répétées), l'auteur indique un procédé basé sur la méthode directe de Liapunov. Ce procédé permet d'imposer aux divers composants de la perturbation des conditions différenciées.

En conclusion, l'auteur analyse la contribution de certains facteurs constructifs à l'amélioration de la stabilité de l'avion en cas de rafales verticales répétées.

Résumé de l'auteur

3174:

Guiraud, J. P. Efforts aérodynamiques non stationnaires sur une aile mince de très faible allongement en déformation. O.N.E.R.A. Publ. no. 95 (1959), 37 pp.

The author considers a flexible, narrow delta wing to be performing simple harmonic oscillations in subsonic flow. In particular, he calculates the work done by the forces in one mode caused by deformations in another. Such expressions are needed in order to calculate the critical velocity of the aerofoil. The method employed is to treat an aerofoil section normal to the free stream, in which case the equation satisfied by the velocity potential, when simple harmonic oscillations are assumed, simplifies to that of Helmholtz in two space variables. Elliptic coordinates in the plane are used and the solution of the differential equation can be written down in terms of Mathieu functions. The pressure at any point of the wing can be obtained from the velocity potential and the forces mentioned above are thus calculated. Some numerical tables are provided which aid computation.

G. A. Lance (Southampton)

3175:
Morton, B. R. On the equilibrium of a stratified layer of fluid. Quart. J. Mech. Appl. Math. **10** (1957), 433–447.

A variational principle of Chandrasekhar [Proc. Cambridge Philos. Soc. **51** (1955), 162–178; MR **16**, 639] is extended to cover fluids possessing both viscosity and thermal conductivity. The extended principle is used to investigate small departures from equilibrium in a freely bounded horizontal fluid layer which is thermally stratified. It is found that viscosity and heat conduction are of equal importance; the initial rate of growth of small disturbances depends on both the Rayleigh and Prandtl numbers. In unstably stratified layers with a linear density gradient, the usual critical Rayleigh number for the onset of convection is obtained. For gradients which are initially non-linear, this critical Rayleigh number is but slightly reduced, and may even be increased in some cases. *H. C. Kranzer* (Garden City, N.Y.)

3176:
Müller, Karl-Heinz. Zur Theorie des Wirbelstrahles. Z. Angew. Math. Mech. **38** (1958), 179–187.

A viscous fluid, confined by a conical wall, circulates with a given constant vorticity about the axis of this cone. The axis is also a line source or sink of constant (perhaps zero) strength. The resulting axially symmetric flow is found explicitly (in terms of hypergeometric functions) by separation of variables in the Navier-Stokes equation. *H. C. Kranzer* (Garden City, N.Y.)

3177:
Saffman, P. G. Exact solutions for the growth of fingers from a flat interface between two fluids in a porous medium or Hele-Shaw cell. Quart. J. Mech. Appl. Math. **12** (1959), 146–150.

"A family of exact solutions of the equations of motion for the two-dimensional unsteady motion of a viscous liquid driven by a fluid of negligible viscosity in a porous medium or Hele-Shaw cell is obtained. The solutions represent the growth of a particular disturbance of a state of uniform motion and its development into a motion in which long fingers of arbitrary width and spacing penetrate the viscous liquid." (Author's summary) *M. H. Protter* (Berkeley, Calif.)

3178:
Tirskii, G. A. An exact solution of the energy equation in a particular case of the motion of a viscous incompressible fluid. J. Appl. Math. Mech. **22** (1958), 777–786 (555–560 Prikl. Mat. Meh.).

In the case of a plane steady flow of a viscous incompressible fluid between two non-parallel planes, the energy equation admits also of an exact solution in terms of Weierstrass elliptic functions, when the Prandtl number is equal to unity or $n(n+1)/12$, n being an integer. The velocity and temperature profiles are investigated for the case of Prandtl number equal to unity in all possible cases. *Y. H. Kuo* (Peking)

3179:
Collins, W. D. Note on a sphere theorem for the axisymmetric Stokes flow of a viscous fluid. Mathematika **5** (1958), 118–121.

Für die schleichende, stationäre Strömung einer zähen Flüssigkeit um eine Kugel wird die Stromfunktion angegeben. Für das unbegrenzte Strömungsfeld weit ausserhalb der Kugel muss nur die Bedingung der Achsensymmetrie erfüllt sein. Die in einer früheren Arbeit noch zusätzlich geforderte Drehungsfreiheit [W. D. Collins, Mathematika **1** (1954), 125–130; MR **16**, 534] ist nicht mehr notwendig. *L. Speidel* (Mülheim)

3180:
Rott, Nicholas. On the viscous core of a line vortex. II. Z. Angew. Math. Phys. **10** (1959), 73–81. (German summary)

[For part I, see same Z. **9b** (1958), 543–553; MR **20** #2947.]

The distribution of temperature is found in a steady axisymmetrical motion in which the components u, v, w of velocity with regard to cylindrical coordinates r, θ, z are given by

$$u = -ar, \quad rv \propto 1-\exp\left(\frac{-ar^2}{2\nu}\right), \quad w = 2az,$$

where a is constant. In the calculation the effects of viscosity and compressibility are assumed small. *W. R. Dean* (London)

3181:
Kuwabara, Sinzi. The forces experienced by randomly distributed parallel circular cylinders or spheres in a viscous flow at small Reynolds numbers. J. Phys. Soc. Japan **14** (1959), 527–532.

In the two-dimensional model the typical circular cylinder is supposed surrounded by a concentric cylinder on which the vorticity is zero and one velocity component is prescribed; the corresponding assumption is made in the three-dimensional case. *W. R. Dean* (London)

3182:
Kuwabara, Sinzi. The forces experienced by a lattice of elliptic cylinders in a uniform flow at small Reynolds numbers. J. Phys. Soc. Japan **14** (1959), 522–527.

Oseen's equations are used in the analysis of the slow two-dimensional motion of viscous liquid past an infinite set of elliptic cylinders; the cylinders are equal and similarly placed, their axis are equidistant parallel straight lines lying in a plane. The general case is also considered. *W. R. Dean* (London)

3183:
Pantchev, St. Corrélation mutuelle de la pression et de la température dans un courant turbulent. C. R. Acad. Bulgare Sci. **12** (1959), 117–120. (Russian. French summary)

On étudie la corrélation mutuelle de la pression et de la température dans un courant homogène, isotrope et stationnaire. Il est prouvé que la fonction de la corrélation mutuelle $B_{PT}(r,\tau)=\overline{pT'}$ satisfait l'équation

$$\kappa\left(\frac{\partial^2 B_{PT}}{\partial r^2}+\frac{2}{r}\frac{\partial B_{PT}}{\partial r}\right) = \pm\frac{\partial B_{TP}}{\partial \tau}.$$

On démontre également l'existence d'un invariant, semblable à celui de Loïtzianski

$$\int_0^\infty r^2 B_{pp} dr = \text{const.}$$

Résumé de l'auteur

3184:

Boumans, A. A. Streaming currents in turbulent flows and metal capillaries. I. Theory (1). Distribution of charge in the liquid. II. Theory (2). Charge transported by the flow of liquid. Physica **23** (1957), 1007–1026; 1027–1037.

3185:

Elder, J. W. The dispersion of marked fluid in turbulent shear flow. J. Fluid Mech. **5** (1959), 544–560.

The author calculates the rate of diffusion of marked fluid in turbulent flow in open channels where the cross-section is (a) semi-circular or (b) rectangular with the width much greater than the depth. An Eulerian analysis is presented using Reynolds' analogy between the transport of mass and momentum, and following G. I. Taylor's theory [Proc. Roy. Soc. London Ser. A. **223** (1954), 446–468] for the same process in a circular pipe. The coefficient of longitudinal diffusion was estimated as $5.9 u^* h$, where u^* is the friction velocity and h the water depth. Experiments on an open flume with dye confirmed this estimate and provided information on the probability density distribution. An indication is given of the modifications required to describe particle diffusion with the same framework, but, although the agreement with experiment is fair, the reviewer's confidence in Reynolds' analogy for this case is strained.

O. M. Phillips (Baltimore, Md.)

3186:

Stratford, B. S. The prediction of separation of the turbulent boundary layer. J. Fluid Mech. **5** (1959), 1–16.

The physical mechanisms underlying the departure of the velocity profile in a rising pressure gradient from its form on a flat plate are discussed. Within the boundary layer the derivative of total pressure along a streamline is equal to that of shear stress normal to the wall. This fact is used, in conjunction with the remarkable assumption that the latter derivative is almost unaffected by the pressure gradient, to relate the velocity and pressure on streamlines not too close to the wall to their values at the pressure minimum.

A second relation between velocity and pressure is obtained both by mixing length theory, and by a neat dimensional analysis (suggested by the reviewer) of the inner velocity profile at the separation point. By joining the two solutions with the normal derivative of velocity, also continuous, a simple criterion for the pressure rise to separation as a function of Reynolds number is found, one constant having been fitted from experiment. High accuracy is not claimed for this result, the author's primary object having been to describe the physics of the flow.

D. A. Spence (Ithaca, N.Y.)

3187:

Ohji, Michio. On the theory of homogeneous axisymmetric turbulence. II. Rep. Res. Inst. Appl. Mech. Kyushu Univ. **6** (1958), 153–171.

This is a continuation of previous work [same Rep. **6** (1958), 63–83; MR **21** #1095] which developed the specification of a field of axisymmetric homogeneous turbulence in terms of second and third order tensors corresponding to the correlation and spectral representations. Explicit relations are here given between the defining scalars of the second-order axisymmetric tensors for $\mathbf{r}$ and $\mathbf{k}$ in homogeneous, incompressible turbulence, and these are related to Chandrasekhar's representation of the velocity field as the curl of a defining vector field. As an example, the analysis is applied to calculate the correlation coefficients after passage of isotropic turbulence through a fine wire gauze.

A. A. Townsend (Cambridge, England)

3188:

Slëzkin, N. A. Theory of cylindrical laminar thermal gas flow. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 109–112. (Russian)

3189:

Aslanov, S. K. The motion of a double wedge-shaped profile at a speed not exceeding that of sound. J. Appl. Math. Mech. **22** (1958), 632–648 (452–464 Prikl. Mat. Meh.).

The hodograph transformation of Chaplygin is used, together with the incompressible flow past a double wedge, to construct a compressible flow. The important feature of the solution is that it has the correct limiting behaviour as the free stream velocity approaches sonic speed.

J. W. Craggs (Newcastle-upon-Tyne)

3190:

Vorob'ev, O. S. An approximate analytical expression for two-dimensional supersonic gas flows. Soviet Physics. Dokl. **122** (3) (1958), 923–927 (778–781 Dokl. Akad. Nauk SSSR).

Wave interactions in two-dimensional steady potential supersonic flow at high Mach number are studied. Hayes noticed that this type of flow is approximately analogous to one-dimensional gas motion. The latter problem was solved already by Riemann, and Love noticed that the solution assumes a relatively simple form for certain ratios of specific heats. The author puts two and two together. A more sophisticated approximation, valid for all ratios of specific heats, is found in J. J. Mahony and R. E. Meyer, Philos. Trans. Roy. Soc. London Ser. A **248** (1956), 467–498 [MR **17**, 913].

R. E. Meyer (Providence, R.I.)

3191:

Cabannes, Henri. Le problème de l'onde de choc attachée stationnaire dans les gaz ionisés. C. R. Acad. Sci. Paris **248** (1959), 3527–3529.

On détermine, pour un gaz doué d'une conductivité électrique infinie, des écoulements stationnaires avec onde de choc attachée à la pointe d'un dièdre; le résultat suivant est obtenu: la présence d'un champ magnétique a pour effet d'augmenter la traînée et de diminuer la température.

Author's summary

3192:
Dobrescu-Purice, Lucia. Géométrisation du phénomène de propagation par ondes, défini par une équation aux dérivées partielles d'ordre II dans l'espace-temps. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 401–414. (Romanian. Russian and French summaries)

L'auteur part de l'équation générale du type hyperbolique et introduit la notion du temps par le phénomène d'évolution et le principe de transversalité. Il y a des relations avec les considérations de Teodorescu. Condition nécessaire et suffisante pour une propagation par ondes épicentriques, au sens de Huygens, est que les équations soient du type hyperbolique normal. *O. Bottema* (Delft)

3193:
Niblett, E. R. The stability of Couette flow in an axial magnetic field. Canad. J. Phys. **36** (1958), 1509–1525.

This paper is an examination of the onset of instability in the Couette flow of an imperfectly electrically conducting, viscous fluid between electrically insulating cylinders in a uniform axial magnetic field. Chandrasekhar [Proc. Roy. Soc. London Ser. A **216** (1953), 293–309; MR **14**, 813] treated this problem for perfectly conducting cylinders by means of a variational principle, and the present author repeats this calculation with the appropriately modified electromagnetic boundary conditions; he uses Chandrasekhar's trial function, which contains one free parameter, but he does not report whether the optimal size of this parameter indicates that the trial function is still a good choice with the new boundary conditions.

The author's result is a table of the critical Taylor number T and critical axial wavelength λ as functions of the Hartman number Q, $0 \leq Q \leq 6400$. For insulating cylinders $T(Q)$ is smaller ($4x$ for large Q) and $\lambda(Q)$ larger ($3x$ for large Q) than for perfectly conducting cylinders.

To verify his results, the author performed experiments on mercury between perspex cylinders inside a coaxial solenoid. These experiments were inconclusive because the cylinders were not longer than the axial wavelengths of the barely unstable modes, and the solenoid was so short that its fringing radial field was 1% of the axial field at the ends of the working volume.

G. Backus (Cambridge, Mass.)

3194:
Iordanskiĭ, S. V. Zemplen's theorem in magnetohydrodynamics. Dokl. Akad. Nauk SSSR **121** (1958), 610–612. (Russian)

The fact that shock waves are necessarily of compressive type is here established for a medium with infinite conductivity in the presence of a magnetic field. It is assumed that

$$(\partial^2 p/\partial V^2)_S > 0, \quad (\partial p/\partial S)_V > 0.$$

The special case of small shocks has been considered by L. D. Landau and E. M. Lifšic [*Élektrodinamika splošnyh sred*, Gosudarstv. Izdat. Tehn.-Teor. Lit. Moscow, 1957; MR **20** #6274]. The proof depends on Hugoniot curves, as in the case of ordinary gas dynamics.

F. V. Atkinson (Canberra)

3195:
Dricot, G.; et Ledoux, P. Remarque sur la théorie des oscillations d'une masse fluide incompressible en présence d'un champ magnétique. Bull. Soc. Roy. Sci. Liège **28** (1959), 115–121. (English summary)

"In the case of an incompressible mass oscillating under its own gravity and a superposed magnetic field, it is shown that the correct boundary condition leads, even if the perturbation of the gravitational field has been neglected in the equation of motion, to a frequency comprising a contribution due to gravity which is usually the dominating one." (Author's summary)

H. Cabannes (Marseille)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 3119, 3127, 3144, 3246.

3196:
Rosenbluth, M. N.; and Kaufman, A. N. Plasma diffusion in a magnetic field. Phys. Rev. (2) **109** (1958), 1–5.

3197:
Debever, R. Connexions métriques des champs électromagnétiques non singuliers dans le vide. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 549–554.

In an earlier paper the author had defined a metric connection in terms of an arbitrary tensor field of rank two. Here he considers the special case of a skew symmetric tensor field and derives a condition, in terms of this connection, for the field to be Maxwellian.

G. Y. Rainich (Notre Dame, Ind.)

3198:
Lebedev, N. N. Distribution of electricity over a thin paraboloid segment. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 513–516. (Russian)

3199:
Bell, P. O. Some applications of differential geometry to the interpretation of physical phenomena. Quart. Appl. Math. **16** (1958), 321–333.

The paper deals with a family F of hypersurfaces in Euclidean n-space. The formulas involve the mean curvature of such a hypersurface, the curvature of their orthogonal trajectory and what the author calls the gauge function, which characterizes how closely the hypersurfaces of F come together at a point. The method involves the consideration of supporting functions and a system of coordinates based on their use, and also, of course, of tensor analysis. Specialization to three-dimensional space leads to interpretation in terms of heat conduction and electric charge distribution on closed conductors.

G. Y. Rainich (Notre Dame, Ind.)

3200:
Vaĭnštein, L. A. Group velocity of damped waves. Ž. Tehn. Fiz. **27** (1957), 2606–2614. (Russian)

3201:
Hauser, Walter. Variational principles for guided electromagnetic waves in anisotropic materials. Quart. Appl. Math. **16** (1958), 259–272.

In this paper Schwinger's variational procedure for the problem of isotropic obstacles in waveguides is extended to the problem of the propagation of waves in a guide which is partially filled with material having tensor electromagnetic properties. The integral formulation of the field vectors in terms of a dyadic Green's function is cast into a variational form whose extremum is shown to be proportional to $(\gamma_0^2-\gamma^2)$, the difference between the square of the propagation constant of the wave in the empty and loaded guides. Suitable normal modes are defined for the expansion of the Green's function, and a first order approximation is calculated for the propagation constant of a wave in a rectangular guide containing an infinite ferrite slab. *R. D. Kodis* (Providence, R.I.)

3202:

Kaufman, H. Circuit response to periodic inputs. Rev. Ci. Lima **59** (1957), 14–20.

3203:

★Cheng, David K. Analysis of linear systems. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1959. xiii+431 pp. $8.50.

This well-organized textbook is intended for advanced undergraduate or beginning graduate students of engineering. It deals primarily with the application of Laplace transform methods to linear systems, principally electric circuits, although a separate chapter on electromechanical analogies and some other occasional analogies are included to increase its relevance for non-electrical engineers.

In view of the purpose for which it is intended, the book quite properly emphasizes the physical understanding of system response at the expense of mathematical generality and rigor. However, this reviewer feels that it is a mistake to avoid the use of complex function theory, particularly in a book for possible use by graduate students dealing with a subject for which a full appreciation comes only through the theory of functions. Furthermore, the use of complex variables would have smoothed a number of rocky paths the author follows in dealing with such topics as the introduction of Laplace transform via the Fourier integral, various inverse transforms, and the Nyquist stability criterion.

The book contains a one-chapter exposition of electric circuit theory, which is probably too sketchy for the uninitiated and unnecessary for others, and a somewhat better review of the bare essentials of linear differential equations. Two relatively recent formalisms are incorporated in the book, namely the signal flow graph for feedback systems and the Z transformation for sampled data systems. This is praiseworthy, although it again indicates the strong bias toward electrical engineering problems. The final chapter is an introduction to the application of Laplace transforms to continuous systems, as exemplified by the transmission line.

The format of the book is excellent, and there is a good selection of examples and problems (with answers).

E. T. Kornhauser (Bristol)

3204:

Ryžov, O. S. On an exact solution of the equations of acoustics. Prikl. Mat. Meh. **21** (1957), 434–437. (Russian)

3205:

Parker, E. N. Newtonian development of the dynamical properties of ionized gases of low density. Phys. Rev. (2) **107** (1957), 924–933.

Dans un gaz ionisé, assez dilué, on néglige les collisions entre particules, la distribution n'étant pas nécessairement isotrope. Le problème des n-corps est alors remplacé par n problèmes à 1-corps. L'état macroscopique du gaz est obtenu en sommant sur les mouvements individuels des particules. En présence d'un champ magnétique, on obtient les équations usuelles de la magnétohydrodynamique. Les densités de courant, la vitesse de masse, et les équations du champ sont obtenues. Les résultats sont appliqués à la discussion de l'équilibre statique.

J. Naze (Marseille)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 3166, 3199.

3206:

Hart, Edward W. Thermodynamics of inhomogeneous systems. II. Phys. Rev. (2) **114** (1959), 27–29.

This paper improves the thermodynamic description of inhomogeneous systems given previously by the author [same Rev. **113** (1959), 412–416; MR **20** #7509] by allowing the energy to depend explicitly on the interaction of the system with the exterior at the boundary.

W. Byers Brown (Manchester)

3207:

Glass, Solomon J.; and Klein, Martin J. Sublimation and the third law of thermodynamics. Physica **25** (1959), 277–280.

"This paper is a discussion of the question: does the existence of non-zero heats of sublimation at absolute zero constitute a violation of the third law of thermodynamics? It is shown that there is no violation, even though the vapor must behave like a "classical" ideal gas and have an entropy per particle proportional to the reciprocal temperature, because the number of particles in the vapour phase vanishes exponentially as absolute zero is approached." (Authors' summary)

W. Byers Brown (Manchester)

3208:

Albertoni, Sergio. Su un problema di propagazione con autovalori per l'equazione del calore. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 206–216.

This paper is concerned with the following rather simplified form

$$\left\{\frac{\partial}{\partial u}-\nabla^2\right\} q(x, y, z, u) = q(x, y, z, u_0)$$

of an equation which arises in the theory of the diffusion of neutrons in a nuclear reactor. Here (x, y, z) are the rectangular cartesian coordinates of a point of the closed region $\bar{D}$, where D is an open region whose boundary σ is a simple closed surface: ∇^2 is Laplace's operator, and u_0

is a given positive value of the independent variable. The solution is required to satisfy the conditions

$$q(x, y, z, u) = 0, \quad \text{when} \quad (x, y, z) \in \sigma, \quad u > 0;$$
$$q(x, y, z, 0) = f(x, y, z), \quad \text{when} \quad (x, y, z) \in D,$$

the function f being continuous in $\bar{D}$.

E. T. Copson (St. Andrews)

3209:

★**Лыков, А. В. (Редактор) Тепло- и массообмен в процессах испарения. [Lykov, A. V. (Editor) Heat and mass exchange in evaporation processes.]** Izdat. Akad. Nauk SSSR, Moscow, 1958. 255 pp. 12.50 rubles.

This is a small booklet for which several Soviet authorities have been asked to write articles contributing to a better understanding of some problems in heat and mass transfer. The aim is primarily didactic and the articles present no particular mathematical interest. Chapter 2 is devoted to papers on the analytical methods which might be of interest to people engaged in applied research and development. It contains a well-written summary on integral transforms by Lykov and Ivanov. It would be interesting to re-analyze the same problems, as well as those in the articles by Alekseeva and Mochalin (on the differential equations of heat conduction and on the application of Dirac's delta function to parabolic equations respectively) in the light of distribution theory.

J. Kestin (Providence, R.I.)

3210:

Caricato, Gaetano. Sulla propagazione stazionaria del calore attraverso un involucro omogeneo limitato da due sfere non concentriche. Rend. Mat. e Appl. (5) **16** (1957), 131–139.

Upper and lower bounds are found for the equilibrium rate of transfer of heat across a homogeneous conducting medium separating two non-concentric spherical surfaces at different temperatures. These bounds are expanded in a power series in the eccentricity ϵ (ratio of distance between centers to difference between radii) and are found to coincide when $\epsilon = 0$.

H. C. Kranzer (Garden City, N.Y.)

QUANTUM MECHANICS

See also 3137, 3259.

3211:

Chang, T. S. Quantization of equations with higher order derivatives. Sci. Record (N.S.) **2** (1958), 253–255.

Two current methods of quantizing theories involving derivatives of order higher than the second are shown to be identical for those problems to which they both apply. The author concludes that in such theories we must abandon hope (a) of finding a positive definite hamiltonian, or (b) of allowing the order of the derivatives to be infinite.

A. J. Coleman (Toronto, Ont.)

3212:

Bernardes, N. Potential energy matrix elements between non-overlapping wave functions. Nuovo Cimento (10) **11** (1959), 628–634. (Italian summary)

Si calcolano gli elementi di matrice dell'energia potenziale di interazione del caso di due funzioni d'onda a simmetria sferica, localizzate e non sovrapposte. Il risultato si esprime in serie di potenze ed è applicabile ad una vasta classe di energie potenziali di forze centrali e di funzioni d'onda non sovrapposte. A titolo illustrativo, si dà una discussione degli elementi di matrici diagonali dell'energia potenziale corrispondenti a una forza di Van der Waals.

Author's summary

3213:

Schulz, G. Kritik des v. Neumannschen Beweises gegen die Kausalität in der Quantenmechanik. Ann. Physik (7) **3** (1959), 94–104.

Von Neumann [*Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin, 1932] gave a proof that causality was inconsistent with quantum mechanics. The paper argues that von Neumann's proof contains an hiatus, to bridge which some such assumption as "All sub-assemblies are experimentally accessible" is needed. Various possibilities opened by this gap in the proof are discussed. Finally Feyerabend's [Z. Physik **145** (1956), 421–423] criticism of the von Neumann proof is reviewed.

C. W. Kilmister (London)

3214:

Okubo, S.; Marshak, R. E.; and Sudarshan, E. C. G. $V-A$ theory and the decay of the Λ hyperon. Phys. Rev. (2) **113** (1959), 944–954.

The decay of the Λ hyperon is studied within the framework of the chirality-invariant four-fermion interaction. Improvements are made on the Born approximation using dispersion theory. The authors solve by function-theoretic methods a pair of singular integral equations, the results being closely related to those of R. Omnes [Nuovo Cimento (10) **8** (1958), 316–326; MR **20** #688].

R. C. MacCamy (Pittsburgh, Pa.)

3215:

Oehme, Reinhard; and Taylor, John G. Proof of dispersion relations for the production of pions by real and virtual photons and for related processes. Phys. Rev. (2) **113** (1959), 371–380.

One of the most important problems of quantum field theory is the derivation of the analytic properties of the scattering amplitudes starting from the so-called "axioms" of the theory.

Such axioms are relativistic invariance, asymptotic conditions, local commutativity of the fields and mass spectrum of the physical particles.

In this paper the validity of fixed momentum transfer dispersion relations is proven for photon and electron production of pions.

The range of values of the momentum transfer for which such dispersion relations can be proven is discussed.

Finally, it is proven that for fixed values of the total energy the absorptive part of the amplitudes are analytic functions of momentum transfer and are regular inside certain ellipses. This makes possible the analytic continuation of the absorptive part into the unphysical region by means of a Legendre polynomial expression.

S. Fubini (Geneva)

3216:

Bialynicki-Birula, I. On the Lee model. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 77–83.

Local quantum field theory is used for two infinitely massive fermions interacting with a boson field. The unrenormalized and renormalized coupling constants are equal. Källén and Pauli [Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 7; MR **17**, 927] have considered the Lee model corresponding to such a theory. This involves removing half of the terms from the interaction Hamiltonian and disturbing the symmetry between creation and annihilation of bosons. Then the lack of symmetry between the self-energy part and the vertex part of diagrams causes the necessity of renormalization of the coupling constant.

C. Strachan (Aberdeen)

3217:

Bollini, C. G. On the quantization of fermion fields with zero mass. Nuovo Cimento (10) **8** (1958), 39–47. (Italian summary)

The formulation of quantum field theory for massless tensor fields given earlier by the author [Nuovo Cimento (10) **6** (1957), 1034–1039; MR **19**, 921] is here extended to fermion fields with zero mass. The fermion field is expressed in terms of simple Dirac spinors. These spinors are independent components of the field and the usual anticommutation relations are imposed on them. The corresponding anticommutation relations of the field components are also given. They are compatible with the supplementary conditions. (From the author's summary)

G. Källén (Lund)

3218:

Van Hove, Léon. Energy corrections and persistent perturbation effects in continuous spectra. Physica **21** (1955), 901–923.

This paper and the following are being reviewed at this late date because of their importance in the quantum theory of fields. In ordinary perturbation theory, as applied, for example, to scattering, it is satisfactory to consider transitions between stationary states of the unperturbed hamiltonian, H, induced by a localized perturber. However, in quantum field theory the interaction involves an integral over all space and also in some crystal problems the perturber is distributed throughout the crystal. In such cases, the stationary states of H have no physical significance and the asymptotic stationary states of the whole system incorporate persistent dissipative and/or "cloud" effects. Hence arises the distinction between "bare" and "clothed" particles ··· the latter providing a better model for physical reality.

This first paper attempts to provide a perturbation theory mathematically adequate to treat this situation. D_l, the diagonal part of the resolvent operator, is defined. The author shows that dissipative or cloud effects occur according as D_l is bounded or unbounded as the complex number l approaches the real axis. In the latter case, he shows that the stationary states of H are not asymptotically stationary states of the total hamiltonian $H+\lambda V$. The chief result of the paper is a formula giving a complete, non-orthogonal set of asymptotically stationary states which incorporate all persistent effects of the perturbation λV. These states are used to express the S matrix as describing transitions between "clothed" states. In the resulting formula the constant of wave renormalization makes a very natural appearance.

A. J. Coleman (Toronto, Ont.)

3219:

Van Hove, Léon. Energy corrections and persistent perturbation effects in continuous spectra. II. The perturbed stationary states. Physica **22** (1956), 343–354.

This continues the paper reviewed above.

The main new result is the explicit determination of a complete set of orthogonal perturbed stationary states (not merely asymptotically stationary as in the first paper). This is used to clarify the physical meaning of the results of the previous paper. The theory is applied to two examples. The first illustrates the fact that van Hove's method gives the usual results for localized perturbations very neatly. The second is one which has been treated differently by other authors, notably T. O. Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317]. The paper concludes with some remarks about renormalization, culminating in the statement "the asymptotically stationary states are the states which must be chosen as basic representation for a more consistent development of the renormalization program."

A. J. Coleman (Toronto, Ont.)

3220:

Mayer, M. E. Extended invariance properties of quantum fields. I. Lagrangian formalism. Nuovo Cimento (10) **11** (1959), 760–770. (Italian summary)

Assuming that the Lagrangian of a quantized field is invariant under an f-parameter Lie-group applied to the set of the components of the field functions, i.e. to the internal degrees of freedom, it is shown that there follow f conservation laws for "charges" of different kind. Since the commutation rules for these charge operators coincide with those of the infinitesimal operators of the group, the charge operators form a representation of these infinitesimal operators in the Hilbert space spanned by the state vectors of the system. It is shown that the internal degrees of freedom of the fields are such that the "internal space" is a finite dimensional Euclidean space, so that the corresponding transformations are orthogonal linear transformations in that space. Finally, the case is investigated where the parameters of the group are functions of ordinary space-time. To ensure invariance under such transformations it becomes necessary to introduce supplementary fields, subject to "gauge transformations of second kind".

P. Roman (Manchester)

3221:

Pandit, L. K. Linear vector spaces with indefinite metric. Nuovo Cimento (10) **11** (1959), supplemento 157–182.

This paper is a review of the properties of linear vector spaces with a metric which is not positive definite. No essentially new results are given, but the topic is presented in a clear and systematic way. Among other things, the paper contains a discussion of the reality of eigenvalues of matrices in such a vector space. In contrast to the case of a positive definite metric, a hermitian matrix does not necessarily have real eigenvalues in a vector space with indefinite metric. Two eigenvectors corresponding to different eigenvalues are not necessarily orthogonal, if the corresponding two eigenvalues are complex conjugate

quantities. At the end of the paper, a few physical applications are discussed as, e.g., the treatment by Gupta and Bleuler of scalar and longitudinal photons in quantum electrodynamics and the Lee model. *G. Källén* (Lund)

3222:

Tanikawa, Yasutaka; and Watanabe, Satosi. Fermi interaction caused by intermediary chiral boson. Phys. Rev. (2) **113** (1959), 1344–1353.

Current renormalization techniques are not applicable to the four-fermion interaction. This paper is an attempt to deduce the successful $V-A$ four-fermion interaction from renormalizable Yukawa couplings. It is necessary to introduce three different spin zero "intermediate bosons" for the beta decay, muon decay and muon capture reactions. The non-locality of the effective interactions is discussed in relation to certain apparent experimental deviations from the predictions of the local $V-A$ interaction. It is claimed that the present introduction of chirality is more fundamental than that of earlier authors. An appendix discusses the concept of chirality in tensor analysis. *E. C. G. Sudarshan* (Cambridge, Mass.)

3223:

Al'perin, M. M. A new method for the linearization of the hamiltonian of a free particle. Dokl. Akad. Nauk SSSR **125** (1959), 1183–1186. (Russian)

The author gives a physical interpretation of the results of a paper by M. S. Livšic [same Dokl. **97** (1954), 399–402; MR **16**, 836]. His conclusion, which he reaches by a perturbation argument, is as follows:

"The behaviour of a particle with non-zero rest mass can be regarded as the result of the interaction of a particle of zero mass with a field of vacuum pairs, the mass playing the role of a constant of the interaction."

F. Smithies (Cambridge, England)

3224:

Gupta, Suraj N. Lorentz covariance of quantum electrodynamics with the indefinite metric. Progr. Theoret. Phys. **21** (1959), 581–584.

It has been questioned whether the use of an indefinite metric for the quantization of the electromagnetic field does not spoil the Lorentz invariance of the state vector. Gupta here shows quite simply that his method of quantization is compatible with Lorentz invariance. The proof makes use of transformation properties, under Lorentz transformations, of quantities which Gupta calls expansors. *H. S. Green* (Adelaide)

3225:

Miyatake, Yoshio. Nonlocal interactions and dispersion relations. Progr. Theoret. Phys. **21** (1959), 562–568.

The author points out that in a field theory where the field operators do not commute for space-like separations, the conventional "reduction formula", where the scattering matrix is expressed with the aid of expectation values of retarded commutators, does not necessarily hold. Consequently, some recent attempts to discuss the modifications of the S-matrix because of non-local or non-causal interactions with the aid of this reduction formula are not reliable. This is illustrated with the aid of perturbation theory computations with non-local interactions of the type considered earlier, e.g., by C. Bloch [Danske Vid. Selsk. Mat.-Fys. Medd. **27** (1952), no. 8].

G. Källén (Lund)

3226:

Medvedev, B. V.; and Polivanov, M. K. On a classical model of an indefinite metric. Soviet Physics. Dokl. **121** (3) (1958), 791–794 (623–626 Dokl. Akad. Nauk SSSR).

A classical field theory is considered in which a complex field $\psi(x)$ interacts with a set of real fields $\varphi_n(x)$. In analogy with a quantum theory with indefinite metric, some of the fields $\varphi_n(x)$ may be associated with negative mass. The requirement that the fields $\varphi_n(x)$ be fictitious in the sense that they cannot exchange energy and momentum with $\psi(x)$ is then shown to lead to an equivalent formulation in which the $\varphi_n(x)$ have been eliminated from the Lagrangian, the resulting theory for $\psi(x)$ being one with non-local interaction. It is suggested that a method due in part to the authors [N. N. Bogoliubov, B. V. Medvedev, and M. K. Polivanov, Nauč. Dokl. Vyšš. Školy, Ser. Fiz.-Mat. no. 1 (1958)] be used in the quantization of such a theory.

A. Klein (Philadelphia, Pa.)

3227:

Kibble, T. W. B.; and Polkinghorne, J. C. Higher order spinor Lagrangians. Nuovo Cimento (10) **8** (1958), 74–83. (Italian summary)

In the $V-A$ form of the Fermi interaction Hamiltonian, the fields are each represented by 2-component spinors only. Feynman and Gell-Mann have suggested that this form of interaction would appear most natural if one considered the electron field, for instance, to be a 2-component spinor satisfying a second order wave equation. However, there are difficulties with this idea. There is no simple Lagrangian which gives the second order wave equation for a 2-component spinor. Alternatively, if one writes a second order, 2-component equation of motion for an electron with the Fermi interaction, then the interaction term must contain either derivatives or 2-component spinors of the opposite kind.

The paper under review considers the problem of finding a Lagrangian for the Feynman-Gell-Mann theory. After rejecting a theory which requires an indefinite metric, a third order Lagrangian is proposed, for which a supplementary condition produces the required asymmetry between the two kinds of 2-component spinor. This theory can be used to limit the interaction to the form A and V, but not to the more special form $A-V$. Thus no satisfactory solution to the problem is found.

The present paper may be compared with another attack on the problem by L. M. Brown [Phys. Rev. (2) **111** (1958), 957–964; MR **20** #6298].

J. C. Taylor (London)

3228:

Nagy, K. L.; and Rzewuski, J. On the equivalence of a certain class of non-local and higher order field theories. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 93–96.

This is a synthesis of work by the authors [Rzewuski, Acta Phys. Polonica **12** (1953), 100–122; MR **16**, 981; Nagy, Nuovo Cimento (10) **10** (1958), 1071–1077; MR **20** #7524] to show that a certain class of non-local field

theories—here a neutral scalar field and a complex scalar field—is equivalent to field theories with higher order Lagrangians. Use is made of Bogoliubov's method [Proceedings 1958 Annual Internation Conference on High Energy Physics at CERN, pp. 129–130, CERN, Geneva, 1958] of two Hilbert spaces, one corresponding to real particles, the other to fictitious particles.

C. Strachan (Aberdeen)

3229:

Federbush, P. G.; and Grisaru, M. T. Representations of symmetry operators. Nuovo Cimento (10) **9** (1958), 890–894.

An explicit representation in terms of the field variables is obtained for the transformation operators of symmetry operations like C, P or T.

The result is first derived in the case of an uncoupled field. The representation is then generalized to coupled fields in the case when the symmetry operators are constants of the motion. *S. Fubini* (Geneva)

3230:

Kapuy, E. Density matrices for wave functions built up from non-orthogonal two-electron orbitals. Acta Phys. Acad. Sci. Hungar. **10** (1959), 125–127.

3231:

Schwartz, Charles. Uses of approximate wave functions. Ann. Physics **6** (1959), 170–177.

A method is described for the calculation of properties of atomic systems to the order of accuracy normally only obtained for the eigenvalues and some examples are discussed. The method has much in common with the methods of Dalgarno and Stewart [see #3232 below].

A. Dalgarno (Belfast)

3232:

Dalgarno, A.; and Stewart, A. L. A perturbation calculation of properties of the helium iso-electronic sequence. Proc. Roy. Soc. London. Ser. A **247** (1958), 245–259.

Perturbation theory is used to calculate various properties of the ground states of helium-like ions. A comparison is made between several methods of calculation. The properties investigated are the magnetic susceptibility, the dipole polarizability, the mass polarization, and the relativistic and radiative corrections to the energy. The methods used enable quantities to be expressed as power series in the inverse of the nuclear charge.

D. J. Thouless (Birmingham)

3233:

Dalgarno, A. Perturbation theory for atomic systems. Proc. Roy. Soc. London. Ser. A **251** (1959), 282–290.

Two procedures for applying the Hartree and Hartree-Fock methods to an atomic system perturbed by a small external field are discussed. The first method is to apply the self-consistency condition to the whole problem, including the perturbation. The second method is to write down a perturbation series starting with the self-consistent solution of the unperturbed problem, and to apply a self-consistency condition to each term in the series. The second method is easier to use, but the first is more accurate. This is shown both by a theoretical argument and by calculating the dipole and quadrupole polarizabilities of helium by the two methods, and comparing the results with experiment.

D. J. Thouless (Birmingham)

3234:

Ford, Kenneth W.; Hill, David L.; Wakano, Masami; and Wheeler, John A. Quantum effects near a barrier maximum. Ann. Physics **7** (1959), 239–258.

In a consideration of the problems of finding the penetrability of a barrier, the splitting of energy levels, and the calculation of scattering phase shifts at energies near the top of a potential barrier the semi-classical JWKB approximation is not valid. Representing the top of the barrier by an inverted oscillator potential, the authors perform an exact calculation of these effects and correlate the exact and approximate solutions. In the analysis the authors also derive a simple expression for $\Gamma(\frac{1}{2}+i\varepsilon)$. *N. S. Wall* (Cambridge, Mass.)

3235:

★Landau, L.; and Smorodinsky, Ya. Lectures on nuclear theory. Consultants Bureau, Inc., 227 West 17th Street, New York 11, N.Y., 1958. 82 pp. $15.00.

The material given here (in an English translation) covers a set of 10 lectures given by Landau (in Russian) in 1954 for experimental physicists. It is concerned largely with the description of phenomenological nuclear theory based on elementary quantum mechanics. The topics covered are nucleon-nucleon scattering, structure of light and heavy nuclei, nuclear reactions (statistical theory and optical model), π-mesons and their interactions with nuclei. The text is well organized and written in an unusually attractive manner. The reader is expected to be acquainted with the elements of modern physics and quantum mechanics, with particular reference to the properties of the angular momentum operators and the vector coupling model. *E. L. Hill* (Minneapolis, Minn.)

3236: ✓

★Landau, L. D.; and Smorodinsky, Ya. Lectures on nuclear theory. Translated from the Russian. Revised ed. Plenum Press, Inc., New York, 1959. vii+108 pp. $5.25.

See the above review.

3237:

★Miller, James; Gerhauser, John M.; and Matsen, F. A. Quantum chemistry integrals and tables. University of Texas Press, Austin, Tex., 1959. 1223 pp. $15.00.

This book provides a useful extension of previous tabulations of molecular integrals by M. Kotani, A. Amemiya, E. Ishiguro and T. Kimura [*Table of molecular integrals*, Maruzen, Tokyo, 1955; MR **19**, 324] and H. Preuss [*Integraltafeln zur Quantenchemie*, vols. 1, 2, Springer-Verlag, Berlin, 1956, 1957; MR **20** #3008]. It may be used to obtain values for all one-centre and two-centre energy and dipole moment integrals involving $1S$, $2S$ and $2p_\sigma$ atomic orbitals of Slater type.

The first section contains general formulae for the integrals in terms of certain auxiliary functions—$A(m;\alpha)$,

$B(n;\beta)$, $G_i(n;\beta)$ and $W_i^0(m,n;\alpha_1\alpha_2)$ in Kotani's notation. These formulae are well adapted for use with internally programmed electronic computers.

The rest of the book is devoted to numerical tables of the above auxiliary functions. Interpolation formulae are given with each table together with an estimate of their accuracy. For the first time (for the W functions) it appears that this accuracy is satisfactory throughout the range of the tables α_1, $\alpha_2 = 1.00\ (0.25)\ 9.00$.

It is unfortunate that integrals involving $2p_\sigma$ orbitals are not considered. However, this is understandable in view of the extent of the tables even with this omission; the table of W functions alone occupies over a thousand pages. *A. C. Hurley* (Melbourne)

3238:

Salam, Abdus; and Ward, J. C. Weak and electromagnetic interactions. Nuovo Cimento (10) **11** (1959), 568–577. (Italian summary)

This paper brings together three main ideas:—(1) That weak interactions are invariant under rotations in a 3-dimensional charge space, Q, not the same as isotopic spin space. (2) That rotations in Q-space should be possible at each point of space-time independently. This implies the existence of a triplet, vector 'B-field' analogous to the isotopic vector B-field of Yang, Mills and Shaw. (3) That two of the components of the B-field (the third being the electromagnetic field) should mediate in the weak interactions.

The self interactions of the B-field correspond to the electrodynamics of a charged vector meson, with just the right Pauli magnetic moment to make it renormalizable. The Q-space symmetry is broken by the 2-component property of the neutrino, thus assigning a special place to the electromagnetic field and allowing the charged B-fields to have an (apparently very large) mass.

No attempt is made in this paper to derive selection rules for the weak interactions. *J. C. Taylor* (London)

3239:

Walecka, J. D. Spectral functions in the static theory. Nuovo Cimento (10) **11** (1959), 821–836. (Italian summary)

The electromagnetic form factors of the nucleon are discussed in the framework of the static theory using the method of the spectral representations.

The same method is applied to study the corrections due to the recoil of the nucleon. Such corrections turn out to be surprisingly large. The physical reasons for such a breakdown of the static approximation are discussed.

S. Fubini (Geneva)

3240:

Rolov, B. N. Two-photon annihilation of pairs of fermions with spin 3/2. Latvijas PSR Zinātņu Akad. Vēstis **1959**, no. 4 (141), 43–48. (Russian. Latvian summary)

3241:

Hayasi, Takesi; Teruyama, Sôgo; and Sagawa, Takasi. Discrete states of an electron in a one-dimensional periodic field having a dislocated cell. Sci. Rep. Tôhoku Univ. Ser. I **42** (1958), 137–143.

3242:

Gamba, A.; Marshak, R. E.; and Okubo, S. On a symmetry in weak interactions. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 881–885.

"The (Λ, n, p) baryon triplet appears to be the minimum number of strongly interacting particles which are necessary to explain conservation of charge, isotopic spin and strangeness" and this triplet appears quite analogous to the (μ, e, ν) lepton triplet. The consequences of the symmetry principle, that the simultaneous interchange $\Lambda \rightleftarrows \mu$, $n \rightleftarrows e$, $p \rightleftarrows \nu$ leaves the weak interactions invariant, are investigated. A unified formula $Q = I_3 + \frac{1}{2}(S + B - L)$ connecting the charge, isotopic spin, strangeness, baryon number and lepton number is also given.

E. C. G. Sudarshan (Cambridge, Mass.)

3243:

Tzou, Kuo-Hsien. Sur l'invariance de chiralité des interactions mésons π-nucléons. C. R. Acad. Sci. Paris **248** (1959), 1300–1303.

The invariance of strong interactions under the chirality transformation $\psi \to \pm\gamma_5\psi$ is investigated and it is asserted that charge-independence and chirality invariance lead to a parity conserving pseudovector interaction.

E. C. G. Sudarshan (Cambridge, Mass.)

3244:

Tyablikov, S. V. On a variational principle in the problem of many bodies. Dokl. Akad. Nauk SSSR **121** (1958), 250–252. (Russian)

It is shown that the condition that a given solution of Bogoliubov's variational method [Dokl. Akad. Nauk SSSR **119** (1958), 244–246; MR **21** #604] gives a minimum for the ground state energy reduces to requiring that the expression for the second variation of the ground state energy be positive. A comparison is made with Fok's variational method. *D. ter Haar* (Oxford)

3245:

Fukuda, Nobuyuki; Wada, Yasushi; and Otake, Shuichi. Gauge invariance and the Meissner effect. Progr. Theoret. Phys. **21** (1959), 343–353.

A gauge invariant method for introducing the electromagnetic field into the Hamiltonian of a many body system interacting via an arbitrary velocity dependent two body potential is given. This is applied to the simple case of a noninteracting electron gas with an arbitrary energy spectrum. It is shown that whether or not the single particle spectrum has an energy gap no Meissner effect follows. In the case that the spectrum has a gap at the Fermi surface a singularity in the current occurs in the long wave length limit. It is thus concluded that a model without electron-electron correlations cannot account for the Meissner effect.

L. N. Cooper (Providence, R.I.)

3246:

Rickayzen, G. Meissner effect and gauge invariance. Phys. Rev. (2) **111** (1958), 817–821.

In the paper it is attempted to derive a Meissner effect from a Hamiltonian which is strictly gauge invariant. In particular, use is made of Fröhlich's Hamiltonian omitting

Coulomb interactions between electrons. The superconducting properties are derived by the use of Bogoliubov's canonical transformation of the Hamiltonian. A Meissner effect is then calculated by introducing the electromagnetic field in a gauge in which div $A=0$. The Meissner effect and an expression for the current at the absolute zero similar to that of Bardeen, Cooper, and Schrieffer [same Rev. **108** (1957), 1175–1204; MR **20** #2196] is obtained. Even though the starting Hamiltonian is gauge invariant doubts have been raised by Wentzel [Phys. Rev. Letters **2** (1959), 33–34] as to whether the calculational procedures are gauge invariant. In fact, Wentzel shows that a gauge with a longitudinal component in A (i.e., div $A\neq 0$) in Rickayzen's approximation also leads to a nonvanishing longitudinal current component.

H. Statz (Waltham, Mass.)

RELATIVITY

3247:

★**Reichenbach, Hans. The philosophy of space and time.** Translated by M. Reichenbach and J. Freund, with introductory remarks by R. Carnap. Dover Publications, Inc., New York, 1958. xvi+295 pp. $2.00.

[English translation of *Philosophie der Raum-Zeit-Lehre*, de Gruyter, Berlin-Leipzig, 1928], a classic in philosophy of science. It does not address itself to physical or mathematical problems, but attempts a logical analysis of special and general relativity. The relations among mathematical formalism, physical theory, and observational data are discussed, and the rôle of convention is examined. The author's theory of "equivalent descriptions" and "coördinating definitions" is advanced to deal with problems such as that of the geometry of physical space. A definition of "universal force" is offered (roughly, a force which affects all physical objects in the same way and against which there can exist no isolating walls) and the principle of the elimination of universal forces is suggested as a method of effecting a unique choice from among a variety of equivalent descriptions, thus rendering the question of the geometry of physical space a determinate empirical question. Among other important problems discussed: visualization (Kant's pure intuition), relations between topological and metrical properties of space and time, relations between causality and space-time, relations between light-geometry and matter-geometry.

W. Salmon (Providence, R.I.)

3248:

Palacios, Julio. Revision of the theory of relativity. Rev. Acad. Ci. Madrid **51** (1957), 21–101, 165–183, 245–292, 405–427. (Spanish. English summaries)

The author believes, mainly because of the so-called clock paradox [H. Dingle, Nature **177** (1956), 782–784; **178** (1956), 680–681; W. H. McCrea, ibid. **177** (1956), 784–785; **178** (1956), 681–682], that special relativity contains self-contradictions. He therefore proposes to drop the hypothesis that all inertial frames are equivalent, and to replace the restricted Lorentz transformation by the following:

$$(1)\qquad \begin{aligned} x' &= \alpha^{-2}(x-vt);\quad y'=y/\alpha;\quad z'=z/\alpha;\\ t' &= \alpha^{-2}(t-vxc^{-2});\quad \alpha=\sqrt{(1-v^2/c^2)}; \end{aligned}$$

where (x', y', z', t') and (x, y, z, t) are the coordinates of an event referred to frames S' and S, S being a frame at absolute rest, and S' moving relative to it with velocity v. It is claimed that in this theory the clock "paradox" does not arise because the duration, measured by S', of an event which occurs at a fixed place in his frame is equal to the duration of the same event as measured by S. The interval between pairs of events is no longer invariant, but the velocity of light is the same for all observers.

The consequences of adopting (1) are worked out for kinematics, dynamics, optics and electromagnetism in inertial frames. One of the results is that universal constants, such as Planck's constant, depend on the coordinate system.

All the formulae of transformation involve the velocities of moving frames relative to S. For two observers to correlate their observations they must know their absolute velocities. The author proposes an experiment to determine the absolute frame, and it is clear that the performance of this is a matter of urgency to any observers who should wish to use this theory.

In a final chapter the theory is applied to the gravitational field of the sun. Here several matters are obscure to the reviewer. Among them are, first, the assumption that an observer falling freely towards the sun is at rest in absolute space and, secondly, the use of a local, non-integrable transformation over the whole of space and time. The author claims to derive results similar to those of general relativity.

W. B. Bonnor (London)

3249:

Hély, Jean. Modèles et mouvements de particules chargées en relativité générale. C. R. Acad. Sci. Paris **248** (1959), 2962–2964.

The author considers a line element of the form

$$ds^2=\gamma dt^2-\frac{dr^2}{\gamma}-r^2(d\theta^2+\sin^2\theta d\varphi^2),$$

which has the same general form as that of Schwarzschild except that γ is given by one of two relations:

$$\frac{3}{4}(\gamma-1)=r^2\int\frac{E^2dr}{r}-\frac{1}{r}\int E^2r^2dr,$$

$$\frac{3}{4}\frac{\gamma-1}{\epsilon}=r^2\int\frac{Edr}{r^3}-\frac{1}{r}\int Edr,$$

where E is a function of r and ϵ is a constant. These line elements furnish models of a universe which comprises electric phenomena.

G. Y. Rainich (Notre Dame, Ind.)

3250a:

Costa de Beauregard, Olivier. Sur l'effet gravitationnel de spin. C. R. Acad. Sci. Paris **245** (1957), 2199–2201.

3250b:

Costa de Beauregard, Olivier. De la trajectoire d'une particule d'épreuve sans spin dans le champ gravitationnel d'une masse douée de spin. C. R. Acad. Sci. Paris **248** (1959), 67–70.

The first of these papers deals with different generalizations of the relation (introduced originally by Einstein) between the velocity of matter and the curvature tensor.

These generalizations consist, of course, in dropping the assumption of symmetry of certain tensors of rank two. In particular, one generalization considered in this paper replaces the fundamental tensor by one that is not symmetric, and the other does the same to the tensor of matter. The second of these assumptions leads to the recognition that the velocity vector V and the momentum vector U have different directions. Their difference W has space direction and plays an important role. In the second paper it is assumed that both of the above tensors are not symmetric. The operation of raising and lowering of indices is given a physical interpretation and seems to be connected with the appearance of the two vectors V and U. The discussion in the second paper deals also with the laws of motion. It deals essentially with approximation. *G. Y. Rainich* (Notre Dame, Ind.)

3251:

Fourès-Bruhat, Yvonne. Conditions de continuité et équations de choc. C. R. Acad. Sci. Paris **248** (1959), 1782–1784.

The author states that if the metric tensor of a space time is such that its components are continuous functions of the local coordinates, if their derivatives are continuous except possibly across a surface of discontinuity, and if the field equations obtain in the sense of the theory of distributions, then the conditions that must be satisfied by the derivatives of the $g_{\mu\nu}$, obtained by the reviewer [Illinois J. Math. **1** (1957), 370–388; MR **19**, 816], must hold. The generalisation of the Rankine-Hugoniot equations also given by the reviewer are obtained from the Bianchi identities. The latter equations are examined in some detail for the case of a perfect charged fluid.

A. H. Taub (Urbana, Ill.)

3252:

Peres, A. Gravitational motion and radiation. I. Nuovo Cimento (10) **11** (1959), 617–627. (Italian summary)

The field equations are solved by expansion in inverse powers of the velocity of light, imposing the de Donder coordinate conditions. The equations of motion then follow without explicitly integrating round singularities.

C. W. Kilmister (London)

3253:

Peres, A. Gravitational motion and radiation. II. Nuovo Cimento (10) **11** (1959), 644–655. (Italian summary)

The method of the previous paper is specialised to find laws of motion directly, without explicit calculation of the g-field. Radiation effects arise independently in the calculation and cause fifth order corrections to the rest-mass of such a sign that a radiating system of masses gains energy. This paradox is discussed.

C. W. Kilmister (London)

3254:

Ghosh, N. N. On wave solution of field equations in Einstein's unified field theory. Indian J. Phys. **33** (1959), 307–315.

A static non-symmetric tensor field $g_{\mu\nu}$ with 4 symmetric components S_{kk}, S_{mm}, S_{nn}, S_{ll} ($k, m, n, l = 1, 2, 3, 4$) and one anti-symmetric component a_{mn} (the rest being all zeros), involving either the coordinate x_k or the coordinates x_k, x_m is made non-static by changing x_k into $x_k + \varepsilon x_l$ where ε is a constant. On solving the relevant field equations in the 'strong' form it is found that such a wave solution is generally spurious, being transformable into corresponding static solution. It is, however, observed that if the field is not spherically symmetric non-trivial solutions in wave form may be constructed.

Author's summary

3255:

Buchdahl, H. A. Gauge-invariant generalization of field theories with asymmetric fundamental tensor. Quart. J. Math. Oxford Ser. (2) **8** (1957), 89–96.

Weyl's gauge-invariant formalism is joined with Einstein's asymmetric metric and affine connection. The physical significance of the geometry thus constructed is not discussed in this paper. The formal construction is straightforward. *P. G. Bergmann* (New York, N.Y.)

3256:

Buchdahl, H. A. Gauge-invariant generalization of field theories with asymmetric fundamental tensor. II. Quart. J. Math. Oxford Ser. (2) **9** (1958), 257–264.

Continuation of a previous paper [reviewed above]. The author attempts to discuss the possible physical significance of the geometric objects introduced, without arriving at any definite conclusions.

P. G. Bergmann (New York, N.Y.)

3257:

Buchdahl, H. A. On phase-invariant and gauge-invariant spinor analysis. Quart. J. Math. Oxford Ser. (2) **9** (1958), 109–113.

The author describes a modification of covariant differentiation of spinors in which the covariant derivatives of the spin-space metric vanish but in which there remains the possibility of retaining the electromagnetic potential. Unimodular transformations in spin-space are not considered in this paper. As far as it goes, the discussion is straightforward.

P. G. Bergmann (New York, N.Y.)

3258:

Husain, Saiyid Izhar. Sur la propagation des discontinuités du tenseur de courbure en théorie unitaire du champ d'Einstein. C. R. Acad. Sci. Paris **248** (1959), 2965–2967.

Let $[R_{\omega\mu\lambda}{}^{\nu}]$ be the discontinuity of the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ along a hypersurface $\Sigma \equiv f(x) = 0$. Then one has by virtue of the Bianchi identity

$$l_{[t}[R_{\omega\mu]\lambda}{}^{\nu}] = 0 \quad (l_t \overset{\text{def}}{=} \partial_t f).$$

If Σ is tangent to the characteristic cone

$$*g^{(\lambda\omega)} l_\lambda l_\omega = 0$$

[for $*g^{\lambda\omega}$ see Hlavatý, *Geometry of Einstein's unified field theory*, Noordhoff, Groningen, 1958; MR **20** #5067; p. 8], then one proves easily that l_μ is a minimal geodesic of the metric $*g^{(\lambda\omega)}$. Using Bianchi's identity in the non-symmetric case [Hlavatý, loc. cit., p. 129] one finds that

$$2l^t D_{[t}[R_{\omega\mu\lambda}{}^{\nu}] + l_\omega[D_t R^t{}_{\mu\lambda}{}^{\nu}] + l_\omega[D_t R_\mu{}^t{}_\lambda{}^\nu]$$

is congruent to zero modulo $[R_{\omega\mu\gamma}{}^{\delta}]$. Here $l^t = *g^{(t\nu)} l_\nu$ and D_μ denotes the covariant derivative with respect to the metric $*g^{(\lambda\nu)}$. *V. Hlavatý* (Bloomington, Ind.)

3259:

Newman, D. J. Entity-structure. Mathematika **5** (1958), 146–160.

A discussion of the foundations of "Structure theory" [Proc. Roy. Irish Acad. Sect. A **59** (1958), 29–47; MR **21** #620], beginning with a division of the physical world into "parts", with "relations" defined by an "object" passing between them. The resulting formalism is reminiscent of that of the creation and annihilation operators in quantum field theory, but the interpretation is quite different. The algebraic structure now arises naturally, instead of being somewhat forced.

A complete macroscopic interpretation of the symbols, and a discussion of quantum theory, are promised.

C. W. Kilmister (London)

ASTRONOMY

3260:

Nahon, Fernand. Sur l'analyse harmonique des distributions de vitesses radiales. C. R. Acad. Sci. Paris **247** (1958), 1311–1313.

It is supposed that the distribution of the velocities in the galactic plane is independent of the longitude, the density of velocity points being $f(\rho, \alpha)$, where ρ and α represent the magnitude and direction of the velocity component in the galactic plane. The frequency function $g(R, \alpha)$ of the radial velocities of stars with the galactic longitude l is then given by the equation

$$g(R, l) = \int_{-\infty}^{+\infty} f(\rho, \theta + l) dT,$$

where $T = \rho \sin \theta$ denotes the tangential velocity. The problem is to determine $f(\rho, \alpha)$ when $g(R, l)$ is known. From the harmonic analysis of the functions $f(\rho, \alpha)$, $g(R, l)$ and the characteristic function of the latter, the author derives some relations between the three functions, making use of Bessel functions. *E. Lyttkens* (Uppsala)

3261:

Nahon, Fernand. Sur la résolution d'une équation intégrale qui généralise l'équation de Jeans des systèmes stationnaires. C. R. Acad. Sci. Paris **247** (1958), 1437–1440.

The integral equation treated in the paper reviewed above is applied to the movements of the stars at right angles to the galactic plane. A cartesian coordinate system is considered, where the coordinates represent the distance from the galactic plane and the velocity perpendicular to that plane. This component of the motion is supposed to be a simple harmonic one. If polar coordinates are introduced and a proper time unit is chosen, the above-mentioned integral equation is obtained, the parameter l being replaced by the time t. The author gives an expression for the coefficient $f_k(\rho)$ of e^{ikx} in the harmonic analysis. *E. Lyttkens* (Uppsala)

3262:

Kavraĭskaya, K. V. Statistical investigation of the ellipticities of double galaxies. Vestnik Leningrad. Univ. **14** (1959), no. 7, 129–139. (Russian. English summary)

The author investigates the problem of the existence of correlations (i) between the true ellipticities and (ii) between the inclinations of the planes of symmetry with respect to the line of vision of the components of double galaxies, and concludes that the statistical research does not reveal any indication of the existence of such correlations [see also same Vestnik **13** (1958), no. 1, 148–158; no. 13, 16–26]. *E. Leimanis* (Vancouver, B.C.)

3263:

Tuominen, Jaakko. On the vortices postulated in von Weizsäcker's theory of the origin of the solar system. Z. Astrophys. **46** (1958), 88–91.

The motion of a particle in a Weizäcker vortex is dealt with. Starting from the hydrodynamical equations of motion in plane polar coordinates and making use of some expansions of the problem of two bodies, the author derives the equations of motion in terms of the disturbance velocities, i.e., the deviations from the circular motion are considered. A remarkable thing is that the usual procedure of neglecting squares and products of the disturbance velocities and their partial derivatives is not allowed. *E. Lyttkens* (Uppsala)

GEOPHYSICS

3264:

Adachi, Ryuzo. On the singular point of travelling-time curve. Kumamoto J. Sci. Ser. A **3**, 98–99 (1957).

Purely theoretical discussion of possible singularities of the observed arrival-time curve. Conclusion: no double point or a cusp of first kind are possible, but only cusps of second kind. *E. Kogbetliantz* (New York, N.Y.)

3265:

Adachi, Ryuzo. Solutions on the reflection method of the seismic prospecting. (General case when 3 dimensions). Kumamoto J. Sci. Ser. A **3**, 93–97 (1957).

This paper is an extension to the three-dimensional case of the study by the same author of the two-dimensional case [same J. **3** (1957), 1–19, 20–24, 25–31; MR **19**, 818]. The method (based on the hypothesis of a constant and known velocity in the first layer) is the same as before and the final solution contains as before the partial derivatives of the equation of the observed and mapped arrival-time surface.

Small errors in these derivatives will cause large errors in the solution and they cannot be avoided because the experimental values of the differentiated function are known at some points only.

E. Kogbetliantz (New York, N.Y.)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 3054.

3266:

Bousquet, G. H. Histoire de l'économie mathématique jusqu'à Cournot. Metroecon. **10** (1958), 121–135.

The author has made bibliographical research to find out to what extent economists were mathematicians

("used formulae") before Cournot. The author's conclusion is that mathematical economics really began with Cournot.

T. Haavelmo (Oslo)

3267:

Osborne, M. F. M. Brownian motion in the stock market. Operations Res. **7** (1959), 145–173.

It is shown that common-stock prices and the value of money can be regarded as an ensemble of decisions in statistical equilibrium. If $Y=\log_e[P(t+\tau)/P(t)]$, where $P(t)$ is the price of a random stock at time t, then Y is considered as a Brownian motion (Wiener) process.

S. Malmquist (Stockholm)

3268a:

Summers, Robert. A note on Least Squares bias in household expenditure analysis. Econometrica **27** (1959), 121–126.

3268b:

Prais, S. J. A comment. Econometrica **27** (1959), 127–129.

The note criticizes the procedure followed by S. J. Prais and the reviewer in *The analysis of family budgets* [Cambridge Univ. Press, England, 1955] for deriving Engel curves from cross-section data. The author argues that using total expenditure as an independent variable to explain particular expenditure "is equivalent to applying Least Squares estimation equation by equation to a system of simultaneous equations", and hence leads to a bias, for which expressions are derived under certain assumptions. The numerical magnitude of the bias is not indicated. In his reply, S. J. Prais regards Summers' assumptions as arbitrary.

H. S. Houthakker (Stanford, Calif.)

3269:

Hu, Te Chiang; and Prager, W. Network analysis of production smoothing. Naval Res. Logist. Quart. **6** (1959), 17–23.

The authors present an elementary method, based upon successive improvements of an initial policy, for treating production smoothing problems. The method is graphical and requires no algebraic or analytic results. It is thus ideally suited for applications.

R. Bellman (Santa Monica, Calif.)

3270:

Sumitsuji, Osamu. Some elementary researches in the mathematics of life insurance. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 163–198. (German, French and Italian summaries)

Es werden mit verschiedenen Methoden Näherungsformeln hergeleitet für Rentenwerte, Nettoprämien und Reserven. Ferner wird gezeigt, wie bekannte Ungleichungen wie diejenige von Steffensen oder Jensen zur Abschätzung versicherungstechnischer Werte dienen können. Anwendungen der Näherungsformeln auf das Studium des Einflusses von Grundlagenänderungen oder erhöhter Risiken.

W. Saxer (Zürich)

3271:

Nolfi, Padrot. Die Berücksichtigung der Sterblichkeitsverbesserung in der Rentenversicherung nach der Optimalmethode der Spieltheorie. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 29–48. (French, Italian and English summaries)

Der Verf. hat im Jahre 1959 neue technische Grundlagen für Pensionsversicherungen herausgegeben [VZ 1960 der Versicherungskasse der Stadt Zürich: Technische Grundlagen für Pensionsversicherungen]. Er zeigt, wie in diesen Grundlagen mit Hilfe geeigneter Modelle der Spieltheorie die Abnahme der Mortalität für die Kosten der Rentenversicherung berücksichtigt wurde.

W. Saxer (Zürich)

3272:

Stelson, Hugh E. Finding the rate in a continuous annuity. Amer. Math. Monthly **65** (1958), 360–362.

The present value of a continuous annuity is given by the formula $B=Ra=R(1-e^{-n\delta})/\delta$; R=periodic payment, n=number of periodic payments, B=amount of loan, δ=rate of interest (to be determined). By writing $I=R_n-B$=the cost of the loan, and $x=n\delta$,

$$B=\frac{2I}{x}\left(1-\frac{x}{6}+\frac{x^2}{36}-\frac{x^3}{540}-\frac{x^4}{6480}-\cdots\right),$$

and by inversion

$$x=2M\left(1-\frac{M}{3}+\frac{2M^2}{9}-\frac{22M^3}{135}-\frac{7M^4}{135}-\cdots\right);\quad M=\frac{I}{B}$$

An approximate value is

$$x'=\frac{2M}{\sqrt[3]{(1+M)}}\quad\left(\text{error}<\frac{8M^4}{405}\right).$$

By considering convergents of the continued fraction

$$x=\frac{2M}{1}+\frac{M/3}{1}+\frac{M/3}{1}+\frac{2M/15}{1}+\cdots$$

the author obtains the following approximations:

$$x_1=\frac{6M}{3+M}\quad\left(|\text{error}|<\frac{2M^5}{g}\right),$$

$$x_2=\frac{2M(3+M)}{3+2M}\quad\left(|\text{error}|<\frac{4M^4}{135}\right),$$

$$x_3=\frac{6M(15+7M)}{45+36M+2M^2}\quad\left(|\text{error}|<\frac{698}{2025}M^5\right).$$

A very good approximation is

$$\frac{x'+x_2}{2}\quad\left(|\text{error}|<\frac{2M^4}{405}\right).$$

S. C. van Veen (Delft)

3273:

D'Esopo, D. A. A convex programming procedure. Naval Res. Logist. Quart. **6** (1959), 33–42.

A procedure is given for the minimization of a convex function of several variables satisfying linear constraints of a special type. Let S be a rectangular parallelepiped in Euclidean n-space. Let an i-segment be the intersection of S with a line parallel to the ith coordinate axis. It is supposed that the convex, differentiable function ϕ on S has a unique minimum on any i-segment; for $z\in S$, denote by $P_i(z)$ that point on the i-segment through z at which ϕ is minimized; let $P=P_nP_{n-1}\cdots P_1$. It is

shown, principally through the continuity of the transformations P_t, that the sequence $\phi(P^t(z))$ tends to the minimum value of ϕ on S for any "starting" z in S.

P. Wolfe (Santa Monica, Calif.)

3274:

★**Aumann, Robert J. Acceptable points in general cooperative n-person games.** Contributions to the theory of games, Vol. IV, pp. 287–324. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

The author generalizes the definition of domination in a natural way to games considered as played without side payments, and calls an undominated payoff vector an acceptable point. The main theorem is that this definition is equivalent to another definition which depends on the notion of a supergame: an infinite sequence of plays of the given game, having for payoff the Cesaro mean of the sequence of payoffs. A strong equilibrium point is defined essentially as a correlated strategy for the set of all players for which the required Cesaro mean exists with probability 1 and no set of players can make a change which has non-zero probability of being advantageous. Two definitions of this form are given. It is shown that they yield the same set of average payoffs, the set of all acceptable points.

{Reviewer's remarks: This is an impressive and potentially valuable technical contribution. However, this reviewer disagrees strongly with numerous remarks such as "the game G just described does not deserve an acceptable point" (page 320). It is not so easy to dismiss games without acceptable points. For example, any essential strong characteristic function game can be treated in the present context by incorporating side payments among the moves; and no such game has an acceptable point. The present paper is a promising start on the large program suggested by Luce and Raiffa on pages 174–177 of *Games and decisions* [Wiley, New York, 1957; MR **19**, 373]. If it is to lead to a general theory, more concepts must be brought in, e.g., some of those indicated by Luce and Raiffa.}

J. Isbell (Lafayette, Ind.)

3275:

★**Thompson, Dorothea M.; and Thompson, Gerald L. A bibliography of game theory.** Contributions to the theory of games, Vol. IV, pp. 407–453. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

This is a reasonably complete bibliography of about 1000 titles in game theory and in related topics that make substantial use of game theory.

A. W. Tucker (Princeton, N.J.)

3276:

★**Gelbaum, B. R. Symmetric zero-sum n-person games.** Contributions to the theory of games, Vol. IV, pp. 95–109. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

Let v_q denote the characteristic function value of a coalition having q members and let

$$a = \max \{v_q/q | 1 \leq q \leq n\}$$

and $p = \min \{q | v_q/q = a\}$. Intuitively, each member of any "minimax coalition" of p players should receive a payoff of a. Using this lead, the author constructs symmetric (and sometimes finite) solutions for large classes of symmetric n-person zero-sum games. The development rests upon certain special assumptions concerning the characteristic function, but games meeting these conditions are shown to exist.

A. W. Tucker (Princeton, N.J.)

3277:

Hanner, Olof. Mean play of sums of positional games. Pacific J. Math. **9** (1959), 81–99.

Milnor [*Contributions to the theory of games*, vol. II, pp. 291–301, Princeton Univ. Press, 1953; MR **14**, 779] defined a positional 2-person game G as follows: G has two players and a finite set of positions P; for each $p \in P$ and each player i there is a set of possible moves $M_i(p) \subset P$. For each p either both $M_1(p)$ and $M_2(p)$ contain at least one move or they are both vacuous. In the latter case p is called an end position. For each end position the payoff functions are defined. The players start with some position and move alternately until an end position is reached, at which time each player collects his payoff. A position in a sum game $G+G^1$ is a pair $(p, p^1) \in P \times P^1$. A move in $G+G^1$ is a move in one of the games G and G^1 and a pass in the other.

The author gives good strategies for sums of games in terms of the properties of the individual games. They are in general not optimal. However, the difference between what a player gets when playing one of them instead of playing an optimal strategy can be estimated. For the sum of n copies of the same game, this difference is bounded for all n. Hence, in mean this difference is small for large n.

M. Dresher (Pacific Palisades, Calif.)

BIOLOGY AND SOCIOLOGY

3278:

Kerner, Edward H. Further considerations on the statistical mechanics of biological associations. Bull. Math. Biophys. **21** (1959), 217–255.

This is an amplification of the author's work on "A statistical mechanics of interacting biological systems" [same Bull. **19** (1957), 121–146; MR **19**, 374]. "It is shown how the Volterra population dynamics, which underlies the statistical theory, can be based on a variational principle; how the dynamics can be generalized as regards both the behaviour of total populations and migration phenomena; and how many directly observable data, such as amplitudes and frequencies of oscillation of a population, fit into the statistical theory and can test it. Such a test is carried out in some detail using the fox-catch data of Elton [*Animal ecology and evolution*, Clarendon Press, Oxford, 1930], with a clear indication that the theory is capable of comprehending the major statistical properties of population-time curves. A final section sketches an extension of the theory to cover secular variations of external conditions such as temperature of the environment." (Author's summary)

M. E. Wise (Sutton)

3279:

Komatu, Yûsaku; and Nishimiya, Han. Probabilistic investigations on inheritance in consanguineous families. Bull. Tokyo Inst. Tech. Ser. B. **1957**, no. 1, 44 pp.

INFORMATION AND COMMUNICATION THEORY

See also 3048.

3280:

Hatori, Hirohisa. On the distribution of completion times for random communication in the task-oriented group with a special structure. Kōdai Math. Sem. Rep. **9** (1957), 23–33.

The author considers an oriented graph consisting of three directed arcs with ends d and e. One of these is directed from e to d and has l edges. The others are directed from d to e. One has m edges and the other n. The vertices are "individuals". It is supposed that each individual has initially one unit of information. At certain "sending times" each individual sends all the information he has acquired to one other individual chosen at random from those to whom he has a directed edge. When each individual is in possession of all the information the process is complete. The author calculates the distribution of completion times in terms of l, m and n.

W. T. Tutte (Toronto, Ont.)

3281:

Carleson, Lennart. Two remarks on the basic theorems of information theory. Math. Scand. **6** (1958), 175–180.

In the first part it is shown how the fundamental theorems of information theory can be extended to the case of denumerable input and output alphabets if the entropies are finite. In the second part it is proved that the capacity of a channel defined in terms of stationary inputs is equal to its capacity defined in terms of ergodic inputs. This was previously shown by Caregradskiĭ [*Arbeiten zur Informationstheorie II*, pp. 65–77, VEB Deutscher Verlag der Wissenschaften, Berlin, 1958; Teor. Veroyatnost. i Primenen. **3** (1958), 84–96; MR **20** #5704, #5703]. However, the new proof is much simpler, and more elementary, than Caregradskiĭ's: The probability measure of an arbitrary non-ergodic input is so modified that the rate of transmission changes by less than an arbitrarily given positive quantity, and that the input becomes a finite-order Markov chain in which all the transition probabilities are positive, which makes it ergodic.

S. K. Zaremba (Swansea)

3282:

★Панов, Д. Ю. Автоматический перевод. [**Panov, D. Yu. Automatic translation.**] 2nd ed. revised and enlarged. Izdat. Akad. Nauk SSSR, Moscow, 1958. 71 pp. 1.10 rubles.

The first edition was reviewed in MR **19**, 1028. The present edition contains information about certain practical results reached since 1956 and about certain recent discussions of the fundamental principles.

SERVOMECHANISMS AND CONTROL

3283:

Anosov, D. V. On stability of equilibrium states of relay systems. Avtomatika i Telemekhanika **20** (1959), 135–149. (Russian. English summary)

Consider a relay system governed by the differential equation

$$(*)\quad (p^n+a_1p^{n-1}+\cdots+a_n)x + M(p^{n-r}+b_1p^{n-r-1}+\cdots+b_{n-r})\operatorname{sgn} x = 0,$$

where the a's, b's and M are constants, $p=d/dt$ and x is the controlled coordinate. Cypkin [*Teoriya relešnyh sistem avtomatičeskogo regulirovaniya*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **18**, 709] conjectured the following criteria for the stability of the trivial solution of (*). (1) If $r=1$, it is necessary that $M>0$ and the real parts of the zeros of $p^n+a_1p^{n-1}+\cdots+a_n$ be nonpositive, and it is sufficient that $M>0$ and the real parts of the zeros be negative. (2) If $r=2$, to the necessary conditions in (1) is added the condition $a_1\geq b_1$, while to the sufficient conditions is added $a_1>b_1$. (3) If $r\geq 3$ or $M=0$, the trivial solution is unstable. These conditions were proved by Boltyanskii and Pontryagin [*Trudy tret'ego vsesoyuznogo matematičeskogo s"ezda, Moskva, iyun'-iyul' 1956*, tom 1, Izdat. Akad. Nauk SSSR, Moscow, 1956; MR **20** #6973a] and are proved by a simpler method in the present paper.

L. A. Zadeh (Berkeley, Calif.)

3284:

Alimov, Yu. I. Calculation of periodic operation conditions in automatic control relay systems. Avtomat. i Telemeh. **20** (1959), 860–866. (Russian. English summary)

Accurate methods of analyzing periodic operation conditions are applied to automatic control systems which consist of relay part, linear units and of inertialess functional generators when the latter are not included in additional feedbacks.

Examples of using the described method are given. Accurate and approximate calculation results are compared for an extremum control relay system.

Author's summary

3285:

Sobolev, Yu. S. Concerning absolute stability of certain control systems. Avtomat. i Telemeh. **20** (1959), 401–405. (Russian. English summary)

The author solves a problem formulated by Letov [*Ustoĭčivost' nelineĭnyh reguliruemyh sistem*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; translated as *Stability of nonlinear regulating systems* by M. D. Friedman, Newtonville, Mass., 1956; MR **17**, 487; **18**, 181] in which it is required to show that if for each member of a certain class B of nonlinear systems the norm of the solution is a strictly decreasing function of time, then the same is true for a certain class $B^*\supset B$.

L. A. Zadeh (Berkeley, Calif.)

3286:

Konstantinesku, Paul [Constantinescu, Paul]. On the synthesis of networks with relay contacts. $(1-k)$-networks. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 377–392. (Russian)

3287:

Elgot, Calvin C.; and Wright, Jesse B. Quantifier elimination in a problem of logical design. Michigan Math. J. **6** (1959), 65–69.

Es ist üblich, das Verhalten von Schaltkreisen in Nachrichtenverarbeitungsgeräten mit Hilfe der Aussagenlogik zu beschreiben, falls dieses Verhalten von der Zeit unabhängig ist oder zumindest die Zeit in dem betreffenden Zusammenhang irrelevant ist. Es liegt nun nahe, das Verhalten zeitabhängiger Schaltkreise durch Formeln zu beschreiben, die Zeitvariablen quantifiziert enthalten. Dabei werden nur diskrete Mengen von Zeitpunkten zugelassen, die jeweils bei Null beginnend gezählt werden.

In der vorliegenden Arbeit wird gezeigt, dass eine bestimmte einfache Sprache nicht ausreicht, um das Verhalten aller Schaltkreise zu beschreiben. Bei dieser Sprache handelt es sich um den einstelligen Prädikatenkalkül, erweitert um die zweistellige Prädikatenkonstante $\leqq$, einen einstelligen Funktor, der die Nachfolgerfunktion im Bereich der natürlichen Zahlen bezeichnet, und schliesslich die Individuenkonstante 0. Es wird gezeigt, dass das Verhalten des sog. "Zeitzählers mod 2" in dieser Sprache nicht beschrieben werden kann. Dieser Zeitzähler mod 2 hat nur einen Ausgang, der genau dann einen Impuls zeigt, wenn t (Zeit) gerade ist. Der Beweis dieses Satzes erfolgt in der Hauptsache unter Verwendung der Methode der Quantorenelimination, die aus der Theorie der Entscheidbarkeit wohlbekannt ist.

Abschliessend sei noch auf einen Druckfehler aufmerksam gemacht: Auf Seite 67 ist in Formel (2) u_i' zu ersetzen durch u_i'. *H. Kiesow* (Münster)

HISTORY AND BIOGRAPHY

See also 3098, 3266.

3288:

★**Manwell, A. R. Mathematics before Newton.** An Inaugural Lecture given in the University College of Rhodesia and Nyasaland. Oxford University Press, London, 1959. 57 pp. 4s: $0.62.

The book is a short survey of the history of mathematics from the earliest times until the seventeenth century. It begins with an account of what Newton achieved and of his influence on eighteenth-century thought. It then describes mathematics of ancient Egypt, Babylonia and Greece, of India and Arabia; it goes into some detail about European medieval mathematics and dynamics, leading on to the contributions of Galileo and Huygens. {At places this interesting account is too condensed. On p. 49 a $\pi/2$ is wanting in the formula for Wallis's infinite product.} A useful list of references is appended.

H. W. Turnbull (Millom)

3289:

Stamatis, Evangelos. Some remarks on successive approximations by iteration among the ancients. Platon **10** (1958), 321–327. (Greek. English summary)

Further discussion [cf. Prakt. Akad. Athenon **31** (1956), 336–346; MR **20** #2245] of iteration from the time of Plato and Archytas, especially as found in the rule for approximation of a square root as indicated in Heron's Metrica.

3290:

Neugebauer, O. The equivalence of eccentric and epicyclic motion according to Apollonius. Scripta Math. **24** (1959), 5–21.

In Almagest XII,1 Ptolemy describes a method for computing the stationary points for a planet, and he mentions explicitly that his method is based upon a theorem by Apollonius concerning the location on the epicycle of such points. An essential feature in Ptolemy's discussion is a method whereby an epicyclic model can be transformed into an excenter model by using an inversion in a fixed circle, which may serve either as excenter or as epicycle. In the article under review the author points out that in Almagest IV,6 Ptolemy uses a method similar to that in XII,1 for the determination of the radius in the epicycle in the simple lunar model. This fact, which so far seems to have escaped attention, makes the author draw the conclusion that the contents of Almagest IV,6 date back to Apollonius. The author further points out that the theorem by Apollonius which Ptolemy uses for the construction of a table of stationary points most likely was used by Apollonius in quite a different manner, for it is shown that the (mean) radius in the epicycle in a simple planetary model can be found from Apollonius' theorem if in addition a (mean) distance between two consecutive stationary points is known.

The author gives a detailed account with appropriate commentaries of the two chapters from Almagest referred to above.

The interest in the present paper lies in the fact that it gives a new and better understanding of the character of Apollonius' contribution to astronomy. His construction of a geometric model for the motion of the moon and the planets rightly entitles him to be considered the real founder of Greek mathematical astronomy.

O. Schmidt (Copenhagen)

3291:

Freudenthal, Hans. Einige Züge aus der Entwicklung des mathematischen Formalismus. I. Nieuw Arch. Wisk. (3) **7** (1959), 1–19.

3292:

Brun, Viggo. Caspar Wessel et l'introduction géométrique des nombres complexes. Rev. Hist. Sci. Appl. **12** (1959), 19–24.

Some historical remarks, with a short discussion of the pedagogical value of Wessel's geometric treatment of complex numbers.

3293:

★**Fleckenstein, J. O. L'école mathématique Baloise des Bernoullis à l'aube du XVIII^e siècle.** Les Conférences du Palais de la Découverte, Série D. No. 62, Histoire des Sciences. Université de Paris, Paris, 1959. 21 pp.

Historical remarks on the mathematical achievements of the Bernoulli family.

3294:

Melanchthon, Philip. A letter of Philip Melanchthon to the reader. Translated by Marian A. Moore. Isis **50** (1959), 145–150.

A first part of this letter, giving arguments for the study of geometry as one of the liberal arts, is strikingly applicable to the present day.

3295:
★**Whittaker, Edmund. From Euclid to Eddington: A study of conceptions of the external world.** Dover Publications, Inc., New York, 1958. ix+212 pp. Paperbound: $1.35.

An unaltered reprint of the first edition [Cambridge University Press, 1949; MR **11**, 2].

3296:
Tosi, Armida. Matematici dell'Osservatorio Astronomico di Brera. Period. Mat. (4) **37** (1959), 78–85.

3297:
★**Ginzel, F. K. Handbuch der mathematischen und technischen Chronologie: Das Zeitrechnungswesen der Völker. Bde. 1, 2, 3.** Deutscher Buch-Export und -Import, GmbH., Leipzig, 1958. Bd. I. xii+584 pp. (1 insert); Bd. II. viii+597 pp.; Bd. III. vii+445 pp. (1 insert)

An unchanged reprint of the first edition [Leipzig, 1906]. Since the work has been a standard reference since its first appearance, there is no need to review it here. Volume 1 deals with the Babylonians, Egyptians, Mohammedans, Persians, Indians, Southeast Asians, Central Americans, Chinese, and Japanese; Volume 2 with the Jews, primitive peoples of Asia, Australia, Africa and America, and Greeks and Romans of classical times; and Volume 3 with the Macedonians, peoples of Asia Minor, Teutonic and Celtic races, nations of medieval Europe, Russians, Byzantians, Armenians, Coptics, Abyssinians. The work closes with a chapter on modern chronology.

3298:
Herneck, Friedrich. Über eine unveröffentlichte Selbstbiographie Ernst Machs. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **6** (1956/57), 209–220. (Russian, English and French summaries)

3299:
Tenca, Luigi. Attività matematica di Lorenzo Lorenzini. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 292–306.

3300:
Varga, Otto. Tibor Szele (1918–1955). Acta Univ. Debrecen. **2** (1955), 5–7. (Hungarian. English summary)

3301:
Robinet, André. La vocation académicienne de Malebranche. Rev. Hist. Sci. Appl. **12** (1959), 1–18.

A discussion of the scientific work of Malebranche, in mathematics, physics and biology, with a view to showing that his influence here is as important as it is in philosophy and religion.

3302:
★**Peano, Giuseppe. Opere scelte. Vol. III. Geometria e fondamenti, Meccanica razionale, Varie.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1959. vii+470 pp. 5000 Lire.

18 articles on geometry and foundations, 7 articles on rational mechanics, 28 miscellaneous articles.

Of special interest in the first sequence are article no. 99, an extract from a work published under the title "I fondamenti dell'aritmetica nel Formulario del 1898" and article no. 105, "Sui numeri irrazionali, 1899", which contain essentially the complete exposition of Peano's foundations of arithmetic; also article no. 64, "Sui fondamenti della geometria, 1894", which is the final revision of Peano's treatment of the foundations of geometry in terms of the primitive notions of point, segment and motion and article no. 98, "Analisi della teoria dei vettori, 1898", which contains the foundations of geometry in terms of the primitive notions of point, vector and scalar product of two vectors.

Of particular interest in the articles on rational mechanics are the three numbered 79, 80 and 89 which deal with the question of the displacement of the earth's pole and a fourth article, no. 87, on the motion of a pendulum whose length is a linear function of the time.

The volume also includes additions to the bibliography published in Vol. I and a complete chronological list of all the works of Peano published in Vol. I [1957; MR **19**, 827], II [1958; MR **20** #6339], and III.

3303:
★**Hecke, Erich. Mathematische Werke.** Herausgegeben im Auftrage der Akademie der Wissenschaften zu Göttingen. Vandenhoeck & Ruprecht, Göttingen, 1959. 955 pp. (1 plate) DM 70.00.

This publication contains all the mathematical works of Hecke (42 articles) in unchanged form, except his book *Vorlesungen über die Theorie der algebraischen Zahlen* [2te Aufl., Akad. Verlagsges, Leipzig 1954; MR **16**, 571]. It also contains a table of his works arranged according to the journal of original publication, a preface on Hecke's scientific work by C. L. Siegel, and a memorial address by J. Nielsen. At the end is given a list of dissertations written under Hecke's direction or approved by him, and a list (13 pages) of remarks and corrections which are referred to in the text but placed at the end in order to leave Hecke's text unchanged. These latter are the work of B. Schoeneberg, who was in general responsible for the preparation of the volume.

MISCELLANEOUS

See also 3036, 3203.

3304:
★**Weinacht, J. H. Prinzipien zur Lösung mathematischer Probleme.** Friedr. Vieweg & Sohn, Braunschweig, 1958. viii+116 pp. DM 8.80.

The book is intended chiefly for teachers of mathematics and for gifted students in the upper classes of secondary schools and in the first year of university. It will also be of great value to a non-mathematician with a philosophical interest in the subject of heuristics, fundamentally important in all science. 45 examples are discussed in

detail, chiefly from geometry, but also from algebra, Diophantine analysis, complex variable etc. The present book should be compared with G. Polya's *How to solve it* [Princeton Univ. Press, Princeton, N.J., 1948; MR **9**, 488].

3305:
Cuesta, N. On the concept, method, teaching and sources of mathematical analysis. Rev. Mat. Hisp.-Amer. (4) **18** (1958), 87–98, 201–213. (Spanish)

3306:
★Кабанов, Г. И. Мой опыт изготовления и применения пособий по геометрии. [Kabanov, G. I. My experience in the preparation and use of models in geometry.] Gosudarstv. Učpedgiz. Minist. Prosveš. PSFSR, Moscow, 1958. 160 pp. 2.20 rubles.

Guidance concerning models constructed from various materials, glass, paper, wire etc., for the teaching of plane and solid geometry.

3307:
★Mitrinović, D. S. Savremene tendencije u nastavi matematike. [Contemporary trends in the teaching of mathematics.] Matematička Biblioteka, 1. Univerzitet u Beogradu. Belgrade, 1957. 31 pp.

Information, based on the work of the Commission Internationale de l'Enseignement Mathématique, about mathematical education in the U.S.A., the U.S.S.R., France, England, and Germany, with remarks on their common features.

3308:
★Mitrinović, D. S. Metod matematičke indukcije. [The method of mathematical induction.] 2nd ed. revised. Matematička Biblioteka, 4. Univerzitet u Beogradu. Belgrade, 1958. 63 pp.

A book written in a clear style with no mathematical prerequisites. The last chapter gives a short history of the subject.

3309:
★Keldysch, M. W. Repetitorium der elementaren Funktionentheorie. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 76 pp. DM 3.60.

A translation by Renate Helle of Chap. 9, Vol. 2 of the Russian work *Matematika: ee soderžanie, metody i značenie* [Izdat. Akad. Nauk SSSR, Moscow, 1956; MR **19**, 520].

3310:
★Moritz, Robert Edouard. On mathematics and mathematicians. (Formerly titled: Memorabilia mathematica or the philomath's quotation-book.) Dover Publications, Inc., New York, 1958. xiii+410 pp. Paperbound: $1.95.

Unaltered republication of first edition [Macmillan, New York, 1914]; 1140 annotated quotations, with cross-index, from mathematicians and other literati.

3311:
★Gardner, Martin. (Editor) Mathematical puzzles of Sam Loyd. Dover Publications, Inc., New York, 1959. xv+165 pp. $1.00.

A selection from the Cyclopedia of Puzzles, privately printed by Samuel Loyd Jr. in 1914, arranged in sections to which the present editor has given the following names: arithmetic and algebra, probability and game theory, operations research, plane geometry, geometrical dissection, route tracing and topological, counter and sliding block, solid geometry.

3312:
★Turkevich, John; and Turkevich, Ludmilla B. Russian for the scientist. D. Van Nostrand Co., Inc., Toronto-Princeton, N.J.-New York-London, 1959. x+255 pp. $5.95.

A practical text-book for the English-speaking scientist. The emphasis is on chemistry, with selections also from aeronautical engineering, biology, and physics.

AUTHOR INDEX—continued

Derwiduć, L. 2727
Deskins, W. E. 2696
D'Esopo, D. A. 3273
Diananda, P. H. 2841
Diaz, J. B. 2803
Dickinson, D. 2762
Dinculeanu, N. 2908, 2909
Ding, Shia-shi. 2907
Dinghas, A. 2745
Di Noi, S. 2945
Ditkin, V. A. 2872
Dixmier, J. 2705
Dobrescu-Purice, L. 3192
Dobrušina, I. S. 3028
Dolbeault-Lemoine, S. 3009
Dorogovcev, A. Ya. 3066
Drăgan, C. 3116
Dragos, L. 2892
Driscol, G. 3195
Drucker, D. C. 3138
Eckmann, B. 2977
Edwards, R. E. 2864
Egervary, B. 3090
Ehresmann, C. 2896
Ehrmann, H. 2780
Elder, J. W. 3185
Elgot, C. C. 3287
Elianu, J. 2849
El'sgol'c, L. È. 2797, 2798
Eltermann, H. 2613
Engel, W. 2651
Englman, R. 3040
Erdélyi, M. 2694
Erdős, P. 2604, 2628, 2631
Erugin, A. N. 2795
Esséen, M. 2869
Fabian, V. 3063
Fabricius-Bjerre, F. 2987
Farahat, H. K. 2676, 2697
Farnell, A. B. 3091
Fast, G. 2961
Fast, H. = Fasć, G.
Fazekas, F. 3093, 3094
Federbush, P. G. 3229
Federer, W. T. 3072
Fel'dman, N. I. 2640
Fempl, S. 2940a–b
Fer, F. 2771
Ficken, F. A. 2894
Finch, P. D. 3053
Finney, D. J. 3064
Fitman, K. M. 2915
Fleckenstein, J. O. 3293
Fleming, W. H. 2724
Flett, T. M. 2878
Florence, A. L. 3164
Foguel, S. R. 2914
Ford, J. 3134
Ford, K. W. 3234
Fort, M. K., Jr. 2595
Fourès-Bruhat, Y. 3251
Fraïssé, R. 2582
Frame, J. S. 3074
Frank, E. 2728
Franklin, J. N. 3105
Freudenthal, H. 3291
Friedman, H. L. 3126
Fujiwara, K. 2905
Fukuda, N. 3245
Fulton, G. E. 2634
Furuta, Y. 2646
Gask, Y. 3045
Gahov, F. D. 2879
Galkin, V. S. 3125
Gambe, A. 3242
Ganguli, P. L. 2712
Gant, P. 2596
Gardner, M. 3311
Garg, K. M. 2714
Garnir, H. G. 2816, 2817, 2818
Gask, H. 2833
Gautschi, W. 3102
Gelbaum, B. R. 2932, 3276
Gemignani, G. 2610
Gerhauser, J. M. 3237
Germogenova, T. A. 2884
Ghourghiev, G. 2989
Ghosh, N. N. 3023, 3254
Ginzel, F. K. 3297
Glass, S. J. 3207
Glicksberg, I. 2917
Gloden, R.-F. 2855
Gnedenko, B. V. 3054
Gobert, J. 2817
Godeaux, L. 2650
Goląb, S. 2659
Goldberg, K. 2985
Golomb, M. 2777
Goodier, J. N. 3164
Gordevskiĭ, D. Z. 2941
Gordon, B. 2846
Gould, H. W. 2597
Graesser, R. F. 2753
Green, M. S. 2985
Greenspan, D. 3099
Greenstein, D. S. 2865
Grisaru, M. T. 3229
Gross, W. A. 3141
Guiraud, J. P. 3174
Gulmanelli, P. 3112
Gunning, R. C. 2751
Gupta, Hansraj 2636
Gupta, Suraj N. 3224
Gusein-Zade, M. I. 3157
Gutmann, H. 2626
Haber, S. 2692
Hacker, T. 3173
Hájek, J. = Gask, Y.
Halpern, E. 2962
Hanner, O. 3277
Harary, F. 2598, 2986
Hart, E. W. 3206
Hashimoto, J. 2607
Haskind, M. D. = Khaskind, M. D.
Hatori, H. 3280
Hauser, W. 3201
Havlíček, K. 3006
Hawkins, D. 2639, 2630
Hayasi, T. 3241
Hecke, E. 3303
Hedlund, G. A. 2595
Hély, J. 3249
Herneck, F. 3298
Heyting, A. 2588, 2589
Higman, G. 2697
Higuti, I. 3080
Hill, D. L. 3234
Hille, E. 2800
Hilton, P. J. 2977
Hirasawa, Y. 2811
Hochschild, G. 2703
Hodge, P. G., Jr. 3161
Horne, J. G., Jr. 2904
Householder, A. S. 3098
Hsiung, Chuan-Chih 3011
Hu, Sze-Tsen 2980
Hu, Te Chiang 3269
Huet, D. 2831
Huppert, B. 2688
Hurst, R. P. 3103
Husain, Saiyid Ishar 3258
Hussain, S. M. 2835
Ingelstam, L. 2614
Ingraham, R. 3136
Institute of Mathematics, Academia Sinica 3156
Iordanskiĭ, S. V. 3194
Isiwata, T. 2966
Itô, N. 2689
Ivanenko, I. P. 3057
Ivanov, V. V. 2883
Ivlev, D. D. 3162
Jabotinsky, E. 2628
Jensen, E. L. 3052
Jones, E. E. 3168
Kabanov, G. I. 3206
Kabanova, V. 2773
Kabe, D. G. 2847
Kac, I. S. 2775, 2776
Kahane, J.-P. 2861
Kajan, Juraj 2701
Kamalamma, K. N. 2992
Kapuy, E. 3230
Karlin, S. 3058
Karpelevič, F. I. 3013
Karzel, H. 2955
Katznelson, Y. 2881
Kaufman, A. N. 3196
Kaufman, H. 3202
Kavraĭskaya, K. V. 3262
Keldysch, M. W. 3309
Kelisky, R. P. 2755
Kennedy, P. B. 2713
Kenyon, H. 2967
Kerner, E. H. 3278
Kertész, A. 2693
Khaskind, M. D. 3171
Kibble, T. W. B. 3227
Kiefer, J. 3079
Kim, E. I. 2881, 2882
Kimura, T. 2770
Kinoshita, N. 2830
Kirgetov, V. I. 3113
Kis, O. 3104
Klamkin, M. S. 2611
Kleene, S. C. 2586, 2590
Klein, M. 3036
Klein, M. J. 3207
Klimontovich, I. L. 3129
Kneser, M. 2643
Koecher, M. 2749
Kofink, W. 3130
Koizumi, S. 2876
Kolibiarová, B. 2699
Kolin, L. 3090
Komatu, Y. 3279
Konijn, H. S. 3071
Konstantinescu, P. 3286
Kooi, O. 2938
Koosis, P. 2866
Korbukova, L. D. 3149
Koreslioglu, H. 3046, 3048
Korobov, N. M. 2625, 2848
Košelev, A. I. 2824
Koshti, M. E. 2847
Krabbe, G. L. 2911, 2912
Kreisel, G. 2591
Krule, I. S. 2710
Krylov, A. L. 2821, 2822
Ku, Chao-hao 3031
Kucsma, M. 2839
Kulshrestha, P. K. 2658, 2744
Kurepa, G. 2603
Kusnetsoff, L. I. 3122
Kuwabara, S. 3181, 3182
Kuzmak, G. E. 2779
Lakshmikantham, V. 3153
Lamperti, J. 2858
Landau, L. D. 3235, 3236
Latter, R. 2890
Lebedev, N. A. 2734
Lebedev, N. N. 3198
Ledoux, P. 3195
Lee, Shen-ling 2988
Leenders, J. H. 2923
deLeeuw, K. 2917
Lefschetz, S. 2793, 2794
Leger, G. F. 2679
Lehti, R. 2951
Leonova, V. B. 3087
Lesieur, L. 2666, 2666, 2667, 2668, 2936
Levitan, B. M. 2805, 2806
Lewy, H. 2810
Ling, Chih-Bing 3145
Locher-Ernst, L. 2946
Lockwood, E. H. 2596
Loinger, A. 3112
Longo, A. 3050
Loś, J. 2684, 2685
Louhivaara, I. S. 2902a–b
Lovass-Nagy, V. 3090
Lowdenslager, D. B. 2836
Lumer, G. 2927
Lusternik, L. A. = Lyusternik, L. A.
Lyance, V. È. 2913
Lykov, A. V. 3209
Lykova, O. B. 2778
Lyusternik, L. A. 2814, 2820
Magenes, E. 2823
Malkin, J. G. 2791
Malliavin, P. 2851
Malyšev, A. V. 2622
Mamuzić, Z. P. 2594
Mandan, Sahib Ram 2944
Mandelbrojt, S. 2851, 2867
Mangeron, D. 3116
Mangler, K. W. 3172
Manin, Yu. I. 2652
Manwell, A. R. 3288
Marčevskiĭ, M. N. 2612
Marchuk, G. I. 3109
Marciniak, Z. 3159
Marcus, F. 3001
Marcus, S. 2592, 2719
Marczewski, E. 2721
Markus, L. 3015
Marshak, R. E. 3214, 3242
Martin, R. M. 2577
Mascart, H. 2741, 2916
Matheson, J. A. L. 3146
Mateos, F. A. 3237
Matsumoto, M. 3008
Matsushita, S. 2834, 2835
Mayer, M. E. 3220
Mayer, O. 3003
Medvedev, B. V. 3226
Mehdi, M. R. 2897
Melman, N. N. 2732
Meisters, G. H. 2782
Melanchthon, P. 3294
Meligy, A. S. 2758
Melzak, Z. A. 2891
Meyer, B. 2968
Meyer zur Capellen, W. 3118
Mickey, R. 3068
Mihăileanu, N. 2997
Mihailescu, E. 2580
Mihailović, D. 2942a–b
Mikami, M. 3082
Mikusiński, J. 2873
Miller, J. 3103, 3237
Miller, M. 2933
Minčev, M. P. 2632
Mirsky, L. 2662, 2676
Miščenko, E. F. 2796
Mitrinović, D. S. 3207, 3208
Miyao, K. 3142
Miyatake, Y. 3225
Moisil, G. C. 2579
Moreau, J.-J. 3167
Morgenstern, D. 3143
Morimoto, H. 2781
Moritz, R. E. 3310
Morse, A. P. 2967
Morton, B. R. 3175
Moses, L. E. 3081
Mostow, G. D. 2703, 2704
Mukhoti, S. R. 2711
Müller, K.-H. 3176
Mura, T. 2830
Murgescu, V. 3019
Murnaghan, F. D. 2695
Mutô, Y. 3026
Myard, F. 3115
Mysovskih, I. P. 2885
Nagai, T. 3025
Nagano, T. 2706
Nagell, T. 2620, 2621
Nagy, K. L. 3228
Nahon, F. 3260, 3261
Naĭmark, B. M. 2825
Nakae, T. 3010
Nakano, T. 2674
Narayana, T. V. 2634
Nash, W. A. 3151
Neiss, F. 2657
Nersesyan, A. B. 2850
Nettleton, R. E. 2985
Neugebauer, O. 3290
Newman, D. J. 2611, 3259
Niblett, E. R. 3193

Nikolaev, V. F. 2857
Nishimiya, H. 3279
Nishimura, T. 2655
Nitsche, J. C. C. 2990a–c
Nolfi, P. 3271
Northcott, D. G. 2671
Nožička, F. 3000, 3114
Obrechkoff, N. 2639
Oehme, R. 3215
Ohji, M. 3187
Okubo, S. 3214, 3242
O'Neill, B. 2983
Onicescu, O. 3119, 3121
Oniščik, A. L. 2707
Osborn, H. A. 2662
Osborne, M. F. M. 3267
O'Shea, S. 2859
Osserman, R. 2991
Ostrowski, A. 2717, 2718
Otake, S. 3245
Otsuki, T. 3018
Ovsyannikov, L. V. 2829
Pac, Mi-Soo 2894, 2970
Palacios, J. 3248
Pandit, L. K. 3221
Panov, D. Yu. 3282
Panychev, S. 3183
Parasyuk, O. S. 2906
Parizeau, B. 2700
Parker, E. N. 3205
Parodi, M. 2730, 2731
Pârvu, M. P. 2925
Pati, T. 2860
Patterson, H. D. 3076
Pavlyuk, I. A. 2785
Payne, L. E. 2809
Peano, G. 3302
Pearl, M. 2660
Pelesaudi, C. 3117
Penzov, Yu. E. 2998
Peres, A. 3252, 3253
Pérez-Cacho, L. 2618, 2619
Pérez-Cacho, T. = Pérez-Cacho, L.
Perumtti, R. 2809
Perrone, N. 3161
Petresco, J. 2682
Petrescu, Ş. 3007, 3020, 3031
Peyerimhoff, A. 2737
Phelps, R. R. 2900
Phillips, A. W. 3084
Pickert, G. 2948, 2949
Plehotin, A. P. 2764
Polivanov, M. K. 3226
Polkinghorne, J. C. 3227
Ponce, J. 2708
Pontryagin, L. S. 2796
Poprutenko, J. 2722
Postnikov, A. G. 3043
Prager, W. 3269
Prais, S. J. 3268b
Prasad, B. N. 2862
Pratt, J. W. 3038
Prigogine, I. 3135
Pucci, C. 2935
Putnam, H. 2583
Rabinowitz, P. 3111
Rachajsky, B. 2769
Rado, R. 2604
Raghavarao, Damaraju 3077
Ramakanth, J. 3153
Rasulov, M. L. 2808
Rédei, L. 2644
Reichenbach, H. 3247
Reiter, H. 2868
Remes, E. Ya. 2854
Rényi, A. 2637, 2639
Revuz, A. 3042
Rhoades, B. E. 2804
Riabouchinsky, D. P. 2726
Richert, H.-E. 2737
Rickayzen, G. 3246
Ridout, D. 2642
Rooinet, A. 3301
Rodnyanskiĭ, A. M. 2971
Rogers, C. A. 2960
Rogers, H., Jr. 2585
Rolf, H. L. 2606
Rolov, B. N. 3240
Romiti, A. 3148
Roos, J.-E. 2833
Rosati, L. A. 2691
Rosenblum, M. 2927
Rosenbluth, M. N. 3196
Rosenfeld, A. 2692
Rosina, B. A. 2648
Roth, J. P. 3092
Rott, N. 3180
Royden, H. L. 2736
Rozovskiĭ, M. I. 2887
Rubin, H. 3058
Rumiantsev, V. V. 3124
Rumyantsev, V. V. = Rumiantsev, V. V.
Ruscior, Ş. 2999
Russell, D. C. 2844
Rybin, P. P. 2886
Ryžov, O. S. 3204
Rzewuski, J. 3228
Sack, R. A. 3049, 3133
Saeki, T. 3016
Saffman, P. G. 3177
Sagawa, T. 3241
Salam, Abdus 3238
Salem, R. 2856
Salzmann, H. 2950
Samarskiĭ, A. A. 2815
Sándor, I. 3093, 3094
Sard, A. 3089
Sąsiada, E. 2686
Sauer, R. 3110
Saul'ev, V. K. 3106
Scafati, M. 2954
Scc, M. 2953
Ščegol'kov, E. A. 2593
Schafer, R. D. 2677, 2678
Schäffer, J. J. 2801
Scherk, P. 2631
Schinzel, A. 2633
Schmidt, H. A. 2587
Schmidtmayer, J. 3100
Schoenfeld, L. 2725
Schönhofer, A. 2840
Schottlaender, S. 2756
Schröter, K. 2581
Schubart, H. 2743
Schulz, G. 3213
Schwartz, C. 3231
Schwarz, Š. 2700
Scipio, L. A., II. 3163
Séminaire Henri Cartan; 10e année: 1957/1958. Fonctions Automorphes. 2750
Sengupta, H. M. 2712
Shephard, G. C. 2960
Shimada, S. 3060
Shioya, S. 3150
Shukla, U. K. 2714
Siddiqi, J. A. 2862
Siddiqi, Omar Ali 2658
Siegel, C. L. 2748
Sieklucki, K. 2974
Sierpiński, W. 2617, 2939
Silin, V. P. 3127
Silvey, S. D. 3062
Sinclair, A. 2746
Sipa, R. 2757
Sispanov, S. 2784
Slězkin, N. A. 3188
Smale, S. 2984
Smeta, H. B. 3132
Smith, J. H. B. 3172
Smith, W. L. 3065
Smorodinsky, Ya. 3235, 3236
Sobolev, Yu. S. 3285
Sokoloff, G. D. 2888
Soldatov, M. A. 2837
Spampinato, N. 2649
Spitzer, F. 3051
Sprinkle, H. D. 2968
Srivastav, R. P. 2739
Srivastava, R. S. L. 2738
Stamatis, E. 3289
Stegmüller, W. 3037
Steinhaus, H. 2721
Stelson, H. E. 3272
Stewart, A. L. 3232
Stinespring, W. F. 2921
Stokes, A. 2768
Stokey, W. F. 3154
St-Pierre, J. 3059
Stratford, B. S. 3183
Striebel, C. T. 3085
Strubecker, K. 2994
Sudarshan, E. C. G. 3214
Sugawara, M. 2981
Sumitsuji, O. 3270
Summers, R. 3268a
Šutov, E. G. 2698
Švec, A. 3002
Sveĭnikov, A. A. 3086
Svijevschi, V. 3116
Szele, T. 2693
Takács, L. 3056
Takasu, T. 2956
Takizawa, S. 3017
Tamássy, L. 2957, 2958
Tamate, O. 3150
Tanikawa, Y. 3222
Tarnove, I. 3096
Tarski, A. 2575
Tate, R. F. 3065
Taylor, J. G. 3215
Taylor, S. J. 2720
Temko, S. V. 3129
Tenca, L. 3299
Terracini, A. 2995, 2996
Teruyama, S. 3241
Theiler, G. 3070
Thomée, V. 2826
Thompson, D. M. 3275
Thompson, G. L. 3275
Tiablikov, S. V. = Tyablikov, S. V.
Timan, M. F. 2852, 2853
Tirskiĭ, G. A. 3178
Titchmarsh, E. C. 2807
Tolmachev, V. V. 3128
Tolokonnikov, L. A. 3140
Tomiyama, J. 2928
Toomre, A. 3170
Tosi, A. 3296
Toupin, R. A. 3144
Tricomi, F. G. 3101
Tsuji, M. 2742
Tuominen, J. 3263
Turán, P. 2638
Turkevich, J. 3312
Turkevich, L. B. 3312
Tverberg, H. 2615
Tyablikov, S. V. 3128, 3244
Tzou, Kuo-Hsien 3243
Unkelbach, H. 2735
Urabe, M. 2783
Urbanik, K. 2919, 3047
Uzakov, Yu. K. 2893
Valiñtefin, L. A. 3200
Valcovici, V. 3120
Vandiver, H. S. 2645
Van Hove, L. 3127, 3218, 3219
Varga, O. 3300
Veloso, E. 2931
Veloso, M. J. 2931
Vicente Gonçalves, J. 2729
Videnskiĭ, V. S. 2760
Vilenkin, N. Ya. 2877
Vilhelm, V. 3030
Vinogradov, I. M. 2624
Vishik, M. I. = Višik, M. I.
Višik, M. I. 2814, 2820
Vodička, V. 3155
Volosov, V. M. 2786
Vorob'ev, O. S. 3190
Wada, Y. 3245
Wakano, M. 3234
Walecka, J. D. 3239
Walfisz, A. 2623
Wang, Guang-ying 2828
Wang, Hao 2584
Wang, Shou-jen 3044, 3083
Wang, Shu-tang 2969
Ward, J. C. 3238
Washnitzer, G. 2654
Watanabe, S. 3232
Waterman, D. 2715
Weinacht, J. H. 3304
Weinberger, H. F. 2803
Weston, J. D. 2723, 2863, 2871, 2873
Weston, V. H. 2763
Wheeler, J. A. 3234
Whittaker, E. 3295
Whittlesey, E. F. 2976
Wielandt, H. 2688
Wijsman, R. A. 3069
Williamson, J. H. 2672
Wing, G. M. 3108
Wintner, A. 2870
Wolfowitz, J. 3079
Wright, J. B. 3287
Wu, Wen-tsün 3027
Wussing, H. 2690
Wuyts-Torfs, M. 2843
Yacoub, K. R. 2683
Yakabe, I. 2641
Yamabe, H. 2813
Yang, T. B. 3041
Yano, K. 3012
Yeh, Yen-chien 2788, 2789
Yood, B. 2924
Young, G. S. 2716
Yuan', Ošao-Din 3107
Zaidman, S. 2812
Zaripov, R. H. 2880, 2889
Zaripov, R. K. = Zaripov, R. H.
Zelen, M. 3078
Želobenko, D. P. 2920
Žen', De-syun 2792
Zeuli, T. 3166
Ziegler, H. 3160
Ziller, A. 3088
Zinger, A. 3059
Zorowski, C. F. 3095, 3154
Zuhovic'kiĭ, S. I. 3087
Zuser, K. 2840